

Free boundary regularity in the optimal partial transport problem

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Abstract

In the optimal partial transport problem, one is asked to transport a fraction $0 < m \leq \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$ of the mass of $f = f\chi_\Omega$ onto $g = g\chi_\Lambda$ while minimizing a transportation cost. If f and g are bounded away from zero and infinity on strictly convex domains Ω and Λ , respectively, and if the cost is quadratic, then the free boundaries of the active regions are shown to be $C_{loc}^{1,\alpha}$ hypersurfaces up to a singular \mathcal{H}^{n-1} negligible set and away from $\partial(\Omega \cap \Lambda)$. This improves and generalizes a result of Caffarelli and McCann [7] and solves a problem discussed by Figalli [8, Remark 4.15]. If in addition one assumes Ω and Λ to be uniformly convex domains with $C^{1,1}$ boundaries, then we prove that the singular set is \mathcal{H}^{n-2} σ -finite in the general case and \mathcal{H}^{n-2} finite if Ω and Λ are separated by a hyperplane.

1 Introduction

Given two non-negative functions $f, g \in L^1(\mathbb{R}^n)$ and a number $0 < m \leq \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$, the optimal partial transport problem consists of finding an optimal transference plan between f and g with mass m . In this context, a transference plan refers to a non-negative, finite Borel measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ with mass m (i.e. $\gamma(\mathbb{R}^n \times \mathbb{R}^n) = m$) whose first and second marginals are controlled by f and g respectively: for any Borel set $A \subset \mathbb{R}^n$,

$$\gamma(A \times \mathbb{R}^n) \leq \int_A f(x)dx, \quad \gamma(\mathbb{R}^n \times A) \leq \int_A g(x)dx.$$

Let $\Gamma_{\leq}^m(f, g)$ denote the set of transference plans. By an optimal transference plan, we mean a minimizer of

$$\inf_{\gamma \in \Gamma_{\leq}^m(f, g)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y). \quad (1.1)$$

Issues of existence, uniqueness, and regularity of optimal transference plans have recently been addressed by Caffarelli & McCann [7] and Figalli [8], [9]. By standard methods in the calculus of variations, one readily obtains existence of minimizers. However, in general, minimizers of (1.1) are far from unique. To see this, let $f \wedge g := \min\{f, g\}$ and suppose $\mathcal{L}^n(\text{supp}(f \wedge g)) > 0$ (with $\mathcal{L}^n(\cdot) := |\cdot|$ being the Lebesgue measure and $\text{supp}(f \wedge g)$ the support of $f \wedge g$). Pick

$$0 < m < \int_{\mathbb{R}^n} (f \wedge g)(x)dx,$$

and let $h < f \wedge g$ be any function with $\|h\|_{L^1(\mathbb{R}^n)} = m$. Note that the transference plan $\gamma_h := (Id \times Id)_\# h$ is optimal (since its cost is zero). However, to construct this family of examples, one needs $\mathcal{L}^n(\text{supp}\{f \wedge g\}) > 0$. Indeed, under a disjointness assumption on the supports, Caffarelli and McCann [7, Theorem 4.3] prove the existence of two domains $U_m \subset \Omega$, $V_m \subset \Lambda$ and a unique convex function Ψ such that the unique minimizer of (1.1) is $\gamma := (Id \times \nabla \Psi)_\# f \chi_{U_m}$, where $\nabla \Psi$ is the optimal transport between $f \chi_{U_m}$ and $g \chi_{V_m}$ ($\overline{U_m \cap \Omega}$ and $\overline{V_m \cap \Lambda}$ are usually referred to as the active regions). Furthermore, by invoking Caffarelli's regularity theory for the Monge-Ampère equation [3], [4], [5], [6], the authors show that if f and g are supported on strictly convex domains separated by a hyperplane, then higher regularity on the densities implies higher regularity on Ψ in the interior of the active region $\overline{U_m \cap \Omega}$ [7, Theorem 6.2]. Moreover, employing a geometric approach, Caffarelli and McCann prove $\Psi \in C_{loc}^{1,\alpha}(\overline{\Omega \cap U_m} \setminus E)$ [7, Corollary 7.14], where $E \subset \partial\Omega$ is a possible singular set, and since $\nabla \Psi$ gives the direction of the normal to the free boundary $\overline{\partial U_m \cap \Omega}$ [7, Corollary 7.15], they also obtain local $C^{1,\alpha}$ regularity of the free boundary (symmetric arguments imply a similar statement for $\overline{\partial V_m \cap \Omega}$ – the free boundary associated to Λ).

Figalli [8] studies the case in which the disjointness assumption on the supports of the densities is removed. He proves that minimizers to (1.1) are unique for

$$\|f \wedge g\|_{L^1(\mathbb{R}^n)} \leq m \leq \min\{\|f\|_{L^1(\mathbb{R}^n)}, \|g\|_{L^1(\mathbb{R}^n)}\},$$

[8, Proposition 2.2 and Theorem 2.10]. In fact, uniqueness of the partial transport is obtained for a general class of cost functions $c(x, y)$, dealing also with the case in which f and g are densities on a Riemannian manifold and $c(x, y) = d(x, y)^2$, where $d(x, y)$ is the Riemannian distance.

As in the disjoint case, Figalli obtains local interior $C^{0,\alpha}$ regularity of the partial transport (i.e. $\Psi \in C_{loc}^{1,\alpha}(U_m \cap \Omega)$) under some weak assumptions on the densities [8, Theorem 4.8]. However, in sharp contradistinction to the disjoint case, he constructs an example with C^∞ densities for which the partial transport is not C^1 , thereby showing that the interior $C_{loc}^{0,\alpha}$ regularity is in this sense optimal [8, Remark 4.9]. Furthermore, by assuming the densities to be bounded away from zero and infinity on strictly convex domains, he goes on to say that Ψ has a C^1 extension to \mathbb{R}^n , and utilizing that $\nabla \Psi$ gives the direction of the normal to $\overline{\partial U_m \cap \Omega}$ (as in the disjoint case), he also derives local C^1 regularity of the free boundary away from $\partial(\Omega \cap \Lambda)$ [8, Theorems 4.10 & 4.11].

However, the author suggests that it may be possible to adapt the method of Caffarelli and McCann to prove Hölder regularity of the partial transport up to the free boundary [8, Remark 4.15]. As a direct corollary, one would thereby improve the C_{loc}^1 regularity of the free boundaries away from the common region into $C_{loc}^{1,\alpha}$ regularity. The first aim of the present work is to prove this result, see Corollary 3.15. Our method of proof follows the line of reasoning in Caffarelli and McCann [7, Section 7], although new ideas are needed to get around the lack of a separating hyperplane. Indeed, as mentioned earlier, Figalli's

counterexample to C^1 regularity of the transport map in the non-disjoint case shows that the assumption of a separating hyperplane plays a crucial role in the regularity theory of the partial transport. The key part of our proof is the adaptation of the uniform localization lemma [7, Lemma 7.11] (cf. Lemma 3.10). This is achieved by classifying the extreme points of the set Z_{min} which comes up in the course of proving this lemma. Indeed, in the disjoint case, Caffarelli and McCann prove that the extreme points are in $\overline{\Lambda}$; however, this is insufficient to close the argument in the general case. To get around this difficulty, we make use of a theorem established by Figalli [8, Theorem 4.10]. Our method has the added feature of allowing us to identify, in a very specific way, the geometry of the singular set which comes up in the work of Caffarelli and McCann and prove the general uniform localization lemma under assumptions which in the disjoint case turn out to be weaker than the ones found in their work [7, Lemma 7.11] (cf. Remark 3.11).

The second aim of this paper is to prove that away from $\partial(\Omega \cap \Lambda)$, the free boundary intersects the fixed boundary in a $C^{1,\alpha}$ way up to a “small” singular set. In the disjoint case, Caffarelli and McCann discovered that this set consists of nontransverse intersection points of fixed with free boundary and points that map to non-locally convex parts of the path-connected target region. Therefore, even in this case, one may not directly apply the implicit function theorem to obtain an estimate on its Hausdorff dimension. However, we exploit the geometry in the uniform localization lemma to prove that in addition to the above description, nontransverse singular points also have the property that when one shoots rays to infinity emanating from these points and in the direction of the normal to the boundary, the half-lines that are generated intersect the closure of the target region only along its boundary (see e.g. Figure 1 and the set X_s in Lemma 3.10). It turns out that this geometry is sufficient to connect the singular set with projections of convex sets onto other convex sets and prove a corresponding rectifiability result; this is the content of Proposition 4.3.

Mathematically, the previous discussion takes the following form: if the supports of the densities are separated by a hyperplane, then as previously mentioned, Caffarelli and McCann prove that $\Psi \in C_{loc}^{1,\alpha}(\overline{\Omega \cap U_m} \setminus E)$, where $E \subset \partial\Omega$ is a closed set [7, Corollary 7.15]. We generalize an improvement of this result to the non-disjoint case. Indeed, our result states that there exists a closed set $\tilde{E} \subset \partial\Omega \cup \partial(\Omega \cap \Lambda)$ for which $\Psi \in C_{loc}^{1,\alpha}(\overline{\Omega \cap U_m} \setminus \tilde{E})$, and if $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\tilde{E} \subset E$ (see Corollary 3.13 and Remark 3.14). Moreover, thanks to the general uniform localization lemma (Lemma 3.10), we are able to identify the set \tilde{E} explicitly in terms of the geometry of Ω and Λ ; using this information we prove, via density estimates, that the free boundary $\partial U_m \cap \overline{\Omega}$ is a $C_{loc}^{1,\alpha}$ hypersurface up to an \mathcal{H}^{n-1} negligible set when the supports are strictly convex (see Proposition 4.1 and Corollary 4.12). If in addition one assumes the supports to be uniformly convex with $C^{1,1}$ boundaries, the singular set for the free boundaries is shown to be relatively closed (away from the common region $\Omega \cap \Lambda$) and \mathcal{H}^{n-2} σ -finite in the general case and compact with \mathcal{H}^{n-2} finite measure in the disjoint case; this is the content of Theorem 4.10.

The paper is organized as follows: in Section 2, we fix some notation and introduce relevant ideas from the literature which will be useful in our analysis. Section 3 is devoted to the $C_{loc}^{0,\alpha}$ regularity theory of the partial transport up to the free boundary; indeed, in this section we utilize the method of Caffarelli and McCann [7, Section 7] to solve the problem mentioned by Figalli [8, Remark 4.15]. Section 4 deals with the Hausdorff dimension of the singular set, and Section 5 discusses several open problems.

2 Preliminaries

In this section, we will fix the notation for the remainder of the paper and state some of the relevant theorems from the literature.

2.1 Notation

Definition 2.1. *Given $\Omega \subset \mathbb{R}^n$ and a convex set $\Lambda \subset \mathbb{R}^n$, we denote the orthogonal projection of Ω onto Λ by $P_\Lambda(\Omega)$.*

Note that in the special case when $\Omega \cap \Lambda = \emptyset$, $P_\Lambda(\Omega) \subset \partial\Lambda$. Hence, we understand $\partial P_\Lambda(\Omega)$ to be the boundary of $P_\Lambda(\Omega)$ seen as a subset of $\partial\Lambda$. In other words, $P_\Lambda(\Omega)$ is a manifold with boundary, and we denote the boundary by $\partial P_\Lambda(\Omega)$. In the general case, $\partial(P_\Lambda(\Omega) \cap \partial\Lambda)$ is defined in a similar way.

Definition 2.2. *Given a C^1 set Λ , we denote the tangent space of Λ at a point $y \in \partial\Lambda$ by $\mathbb{T}_y\Lambda$. Similar notation will be used if the set is Lipschitz.*

Definition 2.3. *Given an $(m-1)$ -plane π in \mathbb{R}^m , we denote a general cone with respect to π by*

$$C_\alpha(\pi) := \{z \in \mathbb{R}^m : \alpha|P_\pi(z)| < P_{\pi^\perp}(z)\},$$

where $\pi \oplus \pi^\perp = \mathbb{R}^m$, $\alpha > 0$, and $P_\pi(z)$ & $P_{\pi^\perp}(z)$ are the orthogonal projections of $z \in \mathbb{R}^m$ onto π and π^\perp , respectively.

Definition 2.4. *Given a convex function Ψ , we denote its corresponding Monge-Ampère measure by*

$$M_\Psi(B) := \mathcal{L}^n(\partial\Psi(B)),$$

where $B \subset \mathbb{R}^n$ is an arbitrary Borel set and $\partial\Psi$ is the sub-differential of Ψ .

Definition 2.5. *For a convex body Z , $t \cdot Z$ denotes the dilation of Z around its barycenter z (center of mass with respect to Lebesgue measure) by a factor $t \geq 0$:*

$$t \cdot Z := (1-t)z + tZ.$$

Definition 2.6. A Radon measure μ on \mathbb{R}^n doubles affinely on $X \subset \mathbb{R}^n$ if there exists $C > 0$ such that each point $x \in X$ has a neighborhood $N_x \subset \mathbb{R}^n$ such that each convex body $Z \subset N_x$ with barycenter in X satisfies $\mu[Z] \leq C\mu[\frac{1}{2} \cdot Z]$. The constant C is called the doubling constant of μ on X , and N_x is referred to as the doubling neighborhood of μ around x .

Definition 2.7. Given $\epsilon > 0$ and a convex function Ψ , we will denote the ϵ centered affine section of Ψ at a locally convex point $z \in \text{dom } \Psi$ (i.e. the domain of Ψ) by

$$Z_\epsilon(z) := Z_\epsilon^\Psi(z) = \{x \in \mathbb{R}^n : \Psi(x) < \epsilon + \Psi(z) + \langle \nu_\epsilon, x - z \rangle\},$$

where $\nu_\epsilon \in \mathbb{R}^n$ is uniquely chosen so that z is the barycenter of $Z_\epsilon(z)$ (see [7, Theorem A.7 and Lemma A.8] and [4]).

Definition 2.8. Fix $p \geq 2$ and a domain $\Omega \subset \mathbb{R}^n$. A locally Lipschitz function $\Psi : \Omega \rightarrow \mathbb{R}$ is p -uniformly convex on Ω if there exists $C > 0$ such that all points of differentiability $x, x' \in \Omega \cap \text{dom } \nabla \Psi$ satisfy

$$\langle \nabla \Psi(x) - \nabla \Psi(x'), x - x' \rangle \geq C|x - x'|^p,$$

where $\text{dom } \nabla \Psi$ is the domain of $\nabla \Psi$.

For an arbitrary convex function Ψ , we recall that its *Legendre transform* is the convex function

$$\Psi^*(y) := \sup_{x \in \mathbb{R}^n} (x \cdot y - \Psi(x)). \quad (2.1)$$

Remark 2.9. As mentioned in [7, Remark 7.10], if a convex function Ψ is p -uniformly convex on $\Omega \subset \text{dom } \Psi$, then $\Psi^* \in C^{1, \frac{1}{p-1}}(\partial \Psi(\Omega))$.

Definition 2.10. Let $Z \subset \mathbb{R}^n$ be a closed convex set. A point $p \in Z$ is said to be *exposed* if some hyperplane touches Z only at p .

Definition 2.11. Let $Z \subset \mathbb{R}^n$ be a closed convex set. A point $p \in Z$ is said to be *extreme* if whenever $p = (1 - \lambda)p_0 + \lambda p_1$ with $\lambda \in (0, 1)$, then $p_0 = p_1$.

2.2 Basic Setup

Given two non-negative, compactly supported functions $f, g \in L^1(\mathbb{R}^n)$, we let

$$\Omega := \{f > 0\} \quad \text{and} \quad \Lambda := \{g > 0\},$$

so that $\Omega \cap \Lambda = \{f \wedge g > 0\}$. We will always assume m to satisfy:

$$\|f \wedge g\|_{L^1(\mathbb{R}^n)} \leq m \leq \min\{\|f\|_{L^1(\mathbb{R}^n)}, \|g\|_{L^1(\mathbb{R}^n)}\}.$$

By the results of Figalli [8, Section 2], we know that there exists a convex function Ψ_m and non-negative functions f_m, g_m for which

$$\gamma_m := (Id \times \nabla \Psi_m)_\# f_m = (\nabla \Psi_m^* \times Id)_\# g_m,$$

is the solution of (1.1) and $\nabla \Psi_m \# f_m = g_m$ (see [8, Theorem 2.3]).

Figalli refers to Ψ_m as the *Brenier solution* to the Monge-Ampère equation

$$\det(D^2 \Psi_m)(x) = \frac{f_m(x)}{g_m(\nabla \Psi_m(x))},$$

with $x \in F_m := \text{set of density points of } \{f_m > 0\}$, and $\nabla \Psi_m(F_m) \subset G_m := \text{set of density points of } \{g_m > 0\}$. Moreover, following Figalli [8, Remark 3.2], we set

$$U_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x}, \bar{y}) \in \Gamma_m} B_{|\bar{x} - \bar{y}|}(\bar{y}),$$

$$V_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x}, \bar{y}) \in \Gamma_m} B_{|\bar{x} - \bar{y}|}(\bar{x}),$$

where Γ_m is the set

$$(Id \times \nabla \Psi_m)(F_m \cap D_{\nabla \Psi_m}) \cap (\nabla \Psi_m^* \times Id)(G_m \cap D_{\nabla \Psi_m^*}),$$

with $D_{\nabla \Psi_m}$ and $D_{\nabla \Psi_m^*}$ denoting the set of continuity points for $\nabla \Psi_m$ and $\nabla \Psi_m^*$, respectively.

We denote the free boundary associated to f_m by $\overline{\partial U_m \cap \Omega}$ and the free boundary associated to g_m by $\overline{\partial V_m \cap \Lambda}$. They correspond to $\overline{\partial F_m \cap \Omega}$ and $\overline{\partial G_m \cap \Lambda}$, respectively [8, Remark 3.3]. Recall from the introduction that one of the goals in this paper is to study the regularity of the free boundaries away from $\partial(\Omega \cap \Lambda)$. One method of attacking this problem is to first prove regularity results on Ψ_m and then utilize that $\nabla \Psi_m$ gives the direction of the normal to the free boundary $\overline{\partial U_m \cap \Omega}$ (by symmetry and duality, this would also imply a similar result for $\overline{\partial V_m \cap \Lambda}$). Indeed, in the following two theorems, Figalli employs this strategy to obtain local C^1 regularity.

Theorem 2.12. [8, Theorem 4.10] *Suppose f, g are supported on two bounded, open, strictly convex sets $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ respectively, and*

$$\|\log(f(x)/g(y))\|_{L^\infty(\Omega \times \Lambda)} < \infty.$$

Then there exists a convex function $\tilde{\Psi}_m \in C^1(\mathbb{R}^n) \cap C_{loc}^{1,\alpha}(U_m \cap \Omega)$ such that $\tilde{\Psi}_m = \Psi_m$ on $U_m \cap \Omega$, $\nabla \tilde{\Psi}_m(x) = x$ on $\Lambda \setminus \overline{V_m}$, and $\nabla \tilde{\Psi}_m(\mathbb{R}^n) = \overline{\Lambda}$. Moreover, $\nabla \tilde{\Psi}_m : \overline{U_m \cap \Omega} \rightarrow \overline{V_m \cap \Lambda}$ is a homeomorphism (with inverse $\nabla \tilde{\Psi}_m^$).*

Theorem 2.13. [8, Theorem 4.11] Assume the setup in Theorem 2.12. Then $(\partial U_m \cap \Omega) \setminus \partial \Lambda$ is locally a C^1 surface, and the vector $\nabla \tilde{\Psi}_m(x) - x$ is different from zero, and gives the direction of the inward normal to U_m .

Remark 2.14. If $x \in (\partial U_m \cap \partial \Omega) \setminus \partial(\Omega \cap \Lambda)$, then $\nabla \tilde{\Psi}_m(x) \neq x$ and the same argument used to prove Theorem 2.13 shows that $\overline{\partial U_m \cap \Omega}$ is locally C^1 away from $\partial(\Omega \cap \Lambda)$.

In our study, we shall also make frequent use of the fact that free boundary never maps to free boundary. This is summed up in the following proposition [8, Proposition 4.13]:

Proposition 2.15. (Free boundary never maps to free boundary) Assume the setup in Theorem 2.12 and let $\tilde{\Psi}_m$ be the corresponding extension of Ψ_m . Then

- (a) if $x \in \partial U_m \cap \Omega$, then $\nabla \tilde{\Psi}_m(x) \notin \overline{\partial V_m \cap \Lambda}$;
- (b) if $x \in \partial U_m \cap \partial \Omega$, then $\nabla \tilde{\Psi}_m(x) \notin \partial V_m \cap \Lambda$.

Moreover, we will also need the fact that the common region $\Omega \cap \Lambda$ is contained in the active regions [8, Remark 3.4]:

Remark 2.16.

$$\Omega \cap \Lambda \subset U_m \cap \Omega, \quad \Omega \cap \Lambda \subset V_m \cap \Lambda.$$

In order to analyze the singular set for the free boundaries, we recall two more sets which will play a crucial role in the subsequent analysis; cf. [7, Equations (7.1) and (7.2)]. The nonconvex part of the free boundary $\overline{\partial U_m \cap \Omega}$ is the closed set

$$\partial_{nc} U_m := \{x \in \overline{\Omega \cap U_m} : \Omega \cap U_m \text{ fails to be locally convex at } x\}. \quad (2.2)$$

Moreover, the nontransverse intersection points are defined by

$$\partial_{nt} \Omega := \{x \in \partial \Omega \cap \overline{\Omega \cap \partial U_m} : \langle \nabla \tilde{\Psi}_m(x) - x, z - x \rangle \leq 0 \quad \forall z \in \Omega\}. \quad (2.3)$$

By duality, $\partial_{nc} V_m$ and $\partial_{nt} \Lambda$ are similarly defined.

2.3 Tools

Next, we collect several well-known results from the literature of convex analysis and geometric measure theory which will be useful in our subsequent analysis. The following lemma is a slight adaptation of such a result [10, Proposition 10.9]. Its corollary follows by a standard covering argument.

Lemma 2.17. Let $M \subset \mathbb{R}^m$ be compact and suppose π is an $(m-1)$ -dimensional hyperplane. If there exist $\delta > 0$ and $\alpha > 0$ such that for all $x \in M$,

$$(B_\delta(x) \cap M) \cap (x + C_\alpha(\pi)) = \emptyset,$$

then there exist $N \in \mathbb{N}$ and Lipschitz functions $f_i : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^m$ where $i \in \{1, \dots, N\}$, such that $M = \bigcup_{i=1}^N f_i(K_i)$, with $K_i \subset \mathbb{R}^{m-1}$ compact. In particular, $H^{m-1}(M) < \infty$.

Corollary 2.18. *Let $M \subset \mathbb{R}^m$ be compact and suppose that for each $x \in M$, $\pi(x)$ is an $(m-1)$ -dimensional hyperplane. If there exist $\delta > 0$ and $\alpha > 0$ such that for all $x \in M$,*

$$(B_\delta(x) \cap M) \cap (x + C_\alpha(\pi(x))) = \emptyset,$$

then there exist $D \in \mathbb{N}$ and Lipschitz functions $f_i : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^m$ where $i \in \{1, \dots, D\}$, such that $M = \bigcup_{i=1}^D f_i(K_i)$, with $K_i \subset \mathbb{R}^{m-1}$ compact. In particular, $H^{m-1}(M) < \infty$.

The next Lemma quantifies the geometric decay of the sections of an arbitrary convex function whose Monge-Ampère measure satisfies a doubling property (see Definition 2.7). The proof may be found in Caffarelli and McCann [7, Lemma 7.6].

Lemma 2.19. *Given $0 \leq t < \bar{t} \leq 1$ and $C > 0$, there exists $s_0 = s_0(t, \bar{t}, \delta, n) \in (0, 1)$, such that whenever Z_ϵ is a fixed section centered at $z_0 \in X := \text{spt} M_\Psi$ of a convex function $\Psi : \mathbb{R}^n \rightarrow (-\infty, \infty]$ whose Monge-Ampère measure satisfies the doubling condition*

$$M_\Psi[Z_{s\epsilon}(z)] \leq CM_\Psi[\frac{1}{2} \cdot Z_{s\epsilon}(z)]$$

for all $s \in [0, 1]$ and all z in the convex set $X \cap Z_\epsilon(z_0)$, then

$$z \in X \cap t \cdot Z_\epsilon(z_0) \implies Z_{s\epsilon}(z) \subset \bar{t} \cdot Z_\epsilon(z_0), \quad \forall s \leq s_0.$$

Corollary 2.20. *Assuming the setup in Lemma 2.19, we have*

$$Z_{s^k\epsilon}(x) \subset \bar{t}^k \cdot Z_\epsilon(x),$$

for all $s < s_0(0, \bar{t})$, $\bar{t} \in (0, 1)$ and integers $k \geq 0$.

The following theorem of Straszewicz establishes a connection between exposed and extreme points of a closed convex set [11, Theorem 18.6].

Theorem 2.21. *For any closed convex set $Z \subset \mathbb{R}^n$, the set of exposed points of Z is a dense subset of the set of extreme points of Z .*

The next theorem of Blaschke is a classical result which states that a family of convex bodies living in a ball admits a converging subsequence in the Hausdorff topology [12].

Theorem 2.22. *The space of all convex bodies in \mathbb{R}^n is locally compact with respect to the Hausdorff metric.*

3 The $C_{loc}^{1,\alpha}$ regularity theory

In what follows, we apply the method of Caffarelli & McCann [7, Section 7] to derive the $C_{loc}^{1,\alpha}$ regularity of the free boundary away from the common region. Unless otherwise stated, we will always assume the following on the initial data:

Assumption 1: Assume f, g are bounded away from zero and infinity on strictly convex, bounded domains $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$, respectively.

Indeed, this is the main assumption of Theorem 2.12, therefore, whenever we will employ this theorem in the statements of our results, Assumption 1 will be implicit. We start the analysis by identifying the support of the Monge-Ampère measure corresponding to $\tilde{\Psi}_m$. By using an equation from the work of Figalli [8, Equation (4.5)], one may prove this result in a similar manner (in fact, almost verbatim) as was done in Caffarelli and McCann [7, Lemma 7.2]; hence, we omit the details.

Lemma 3.1. *Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then $\tilde{\Psi}_m$ has a Monge-Ampère measure that is absolutely continuous with respect to Lebesgue, and there exist positive constants c, C (depending on the initial data) so that for any Borel set $E \subset \mathbb{R}^n$,*

$$c|E \cap (\Omega \cap U_m)| + |E \cap (\Lambda \setminus V_m)| \leq M_{\tilde{\Psi}_m}(E) \leq C|E \cap (\Omega \cap U_m)| + |E \cap (\Lambda \setminus V_m)|. \quad (3.1)$$

Next, we identify a set on which the Monge-Ampère measure corresponding to the convex function $\tilde{\Psi}_m$ doubles affinely (recall Definition 2.6). This will be useful in quantifying the strict convexity of $\tilde{\Psi}_m$.

Lemma 3.2. *Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then $\tilde{\Psi}_m$ has a Monge-Ampère measure $M_{\tilde{\Psi}_m}$ which doubles affinely (see Definition 2.6) on*

$$X := \overline{\Omega \cap U_m} \setminus \left(\partial_{nc} U_m \cup \left(\overline{\partial V_m \cap \Lambda} \cap \partial(\Omega \cap \Lambda) \right) \right).$$

Moreover, any ball $N_x = B_R(x)$ which has a convex intersection with $\Omega \cap U_m$ and is disjoint from $\Lambda \setminus V_m$ is a doubling neighborhood around x .

Proof. First, since $\partial V_m \cap \Lambda$ does not intersect $U_m \cap \Omega$ (by Remark 2.16), the only place where $\Lambda \setminus V_m$ may possibly intersect $\overline{U_m \cap \Omega}$ is on $\overline{\partial V_m \cap \Lambda} \cap \partial(\Omega \cap \Lambda)$. Now if $x \in X$, then there exists $R = R(x) > 0$ for which $B_R(x) \cap (\Lambda \setminus V_m) = \emptyset$ and $W := \Omega \cap U_m \cap B_R(x)$ is convex. With this in mind, thanks to Lemma 3.1, we may proceed verbatim as in [7, Lemma 7.5] and [4, Lemma 2.3] to prove the result. □

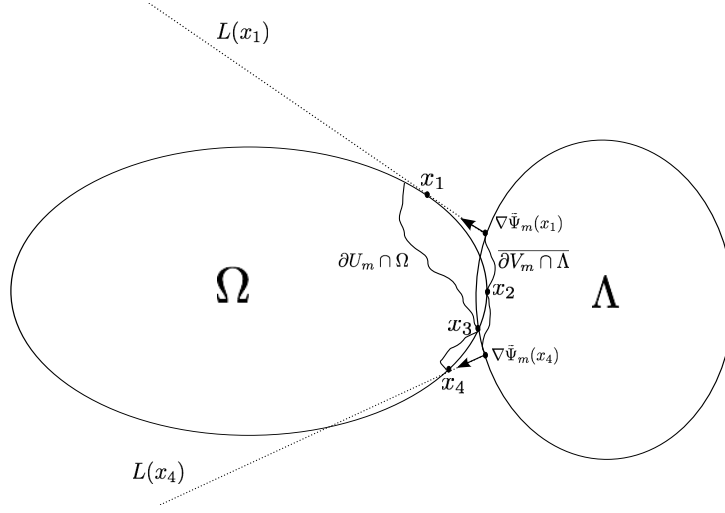


Figure 1: $x_1 \in A_2$; $x_4 \in A_1$; $x_2, x_3 \in S_2$.

Note that the set X from the previous lemma on which $M_{\tilde{\Psi}_m}$ doubles affinely excludes non-locally convex points in $\overline{\Omega \cap U_m}$; since Caffarelli's regularity theory employs the doubling property, and since the active region is not necessarily convex, this suggests the existence of a potential singular set. Indeed, we will now define and prove some topological results of various sets which will naturally come up in the course of our study; these sets will be used to construct candidates for the singular set. Although seemingly technical, they have a very geometric flavor, see Figure 1.

Definition 3.3. (*Components of the singular set*) Let $\tilde{\Psi}_m$ be as in Theorem 2.12. Then, for $x \in \partial(\Omega \cap U_m)$ and $x \neq \nabla \tilde{\Psi}_m(x)$, let

$$L(x) := \left\{ \nabla \tilde{\Psi}_m(x) + \frac{x - \nabla \tilde{\Psi}_m(x)}{|x - \nabla \tilde{\Psi}_m(x)|} t : t \geq 0 \right\};$$

$$K := \left\{ x \in \partial(\Omega \cap U_m) : \nabla \tilde{\Psi}_m(x) \neq x, L(x) \cap \overline{\Omega \cap U_m} \subset \partial(\overline{\Omega \cap U_m}) \right\};$$

The following two sets play a critical role in our study:

$$S_1 := \nabla \tilde{\Psi}_m^{-1}(\partial_{nt} \Lambda) \cap K;$$

$$S_2 := (\partial U_m \cap \partial \Lambda \cap \Omega) \cup (\partial \Omega \cap \partial \Lambda \cap \{\nabla \tilde{\Psi}_m(z) = z\}) \cup (\partial V_m \cap \partial \Omega \cap \Lambda).$$

It will prove useful for us to decompose S_1 into the part which touches the free boundary and the part which is disjoint from the free boundary:

$$A_1 := S_1 \cap \partial U_m;$$

$$A_2 := S_1 \setminus \partial U_m.$$

Remark 3.4. If $x \in S_1 \cap \Omega$, then $x \in \partial U_m \cap \Omega$ and $\nabla \tilde{\Psi}_m(x) \in \partial_{nt} \Lambda \subset \overline{\partial V_m \cap \Lambda}$, a contradiction to Proposition 2.15. Hence, $S_1 \subset \partial \Omega$.

Lemma 3.5. $(\partial U_m \cap \partial \Lambda \cap \overline{\Omega}) \cup (\partial V_m \cap \partial \Omega \cap \overline{\Lambda}) \subset \{\nabla \tilde{\Psi}_m(z) = \nabla \tilde{\Psi}_m^*(z) = z\}$.

Proof. Suppose first that $x \in \partial V_m \cap \partial \Omega \cap \overline{\Lambda}$. Then, since $\partial V_m \cap \partial \Omega \cap \overline{\Lambda} \subset \overline{\Lambda \setminus V_m}$, it follows that $\nabla \tilde{\Psi}_m(x) = x$ by Theorem 2.12. But by Remark 2.16, we know

$$\Omega \cap \Lambda \subset (\Lambda \cap V_m) \cap (\Omega \cap U_m);$$

therefore, $\partial V_m \cap \partial \Omega \cap \overline{\Lambda} \subset \overline{\Lambda \cap V_m \cap \Omega \cap U_m}$, and since $\nabla \tilde{\Psi}_m : \overline{\Omega \cap U_m} \rightarrow \overline{\Lambda \cap V_m}$ is a homeomorphism with inverse $\nabla \tilde{\Psi}_m^*$, we have $\nabla \tilde{\Psi}_m^*(x) = \nabla \tilde{\Psi}_m^*(\nabla \tilde{\Psi}_m(x)) = x$. An entirely symmetric argument yields $\partial U_m \cap \partial \Lambda \cap \overline{\Omega} \subset \{\nabla \tilde{\Psi}_m(z) = \nabla \tilde{\Psi}_m^*(z) = z\}$. \square

Remark 3.6. We note that if $x \in S_2$, then by Lemma 3.5, $\nabla \tilde{\Psi}_m(x) = x$ so $\nabla \tilde{\Psi}_m(S_2) = S_2$.

Lemma 3.7. Let $X_s := S_1 \cup S_2$. Then S_2 and X_s are compact.

Proof. First, we note that $X_s \subset \overline{\Omega \cup \Lambda}$, and since $\overline{\Omega \cup \Lambda}$ is bounded, it suffices to prove that S_2 and X_s are closed. First, we prove the assertion for S_2 : note that by Lemma 3.5,

$$\begin{aligned} \overline{S_2} &\subset (\partial U_m \cap \partial \Lambda \cap \overline{\Omega}) \cup (\partial \Omega \cap \partial \Lambda \cap \{\nabla \tilde{\Psi}_m(z) = z\}) \cup (\partial V_m \cap \partial \Omega \cap \overline{\Lambda}) \\ &\subset \left((\partial U_m \cap \partial \Lambda \cap \overline{\Omega}) \cup (\partial \Omega \cap \partial \Lambda) \cup (\partial V_m \cap \partial \Omega \cap \overline{\Lambda}) \right) \cap \{\nabla \tilde{\Psi}_m(z) = z\} \\ &\subset S_2 \cup (\partial \Omega \cap \partial \Lambda \cap \{\nabla \tilde{\Psi}_m(z) = z\}) \subset S_2. \end{aligned}$$

Next, we show $\overline{S_1} \subset S_1 \cup S_2 = X_s$. Indeed, suppose $\{x_n\} \subset S_1$ with $x_n \rightarrow x \in \overline{\Omega \cup \Lambda}$. Then, as $\partial(\Omega \cap U_m)$ is compact, we have that

$$x \in \partial(\Omega \cap U_m). \tag{3.2}$$

Let $y_n := \nabla \tilde{\Psi}_m(x_n) \in \partial_{nt} \Lambda \subset \partial \Lambda \cap \partial V_m$ so that for all $z \in \Lambda$,

$$\langle \nabla \tilde{\Psi}_m^*(y_n) - y_n, z - y_n \rangle \leq 0.$$

By continuity of $\nabla\tilde{\Psi}_m$ and compactness of $\partial\Lambda \cap \partial V_m$, $y_n \rightarrow \nabla\tilde{\Psi}_m(x) =: y \in \partial\Lambda \cap \partial V_m$, and by continuity of $\nabla\tilde{\Psi}_m^*$ and of the inner product, it follows that for all $z \in \Lambda$,

$$\langle \nabla\tilde{\Psi}_m^*(y) - y, z - y \rangle \leq 0.$$

Hence, $y \in \partial_{nt}\Lambda$ and

$$x = \nabla\tilde{\Psi}_m^*(y) = (\nabla\tilde{\Psi}_m)^{-1}(y) \in \nabla\tilde{\Psi}_m^{-1}(\partial_{nt}\Lambda). \quad (3.3)$$

Let us first assume $\nabla\tilde{\Psi}_m(x) \neq x$. In this case, if there exists $t \geq 0$ such that

$$\nabla\tilde{\Psi}_m(x) + \frac{x - \nabla\tilde{\Psi}_m(x)}{|x - \nabla\tilde{\Psi}_m(x)|}t \in \Omega \cap U_m,$$

then since $\Omega \cap U_m$ is open, for n large enough we will also have

$$\nabla\tilde{\Psi}_m(x_n) + \frac{x_n - \nabla\tilde{\Psi}_m(x_n)}{|x_n - \nabla\tilde{\Psi}_m(x_n)|}t \in \Omega \cap U_m,$$

a contradiction to the fact that $x_n \in K$. Therefore, we obtain that for all $t \geq 0$,

$$\nabla\tilde{\Psi}_m(x) + \frac{x - \nabla\tilde{\Psi}_m(x)}{|x - \nabla\tilde{\Psi}_m(x)|}t \notin \Omega \cap U_m;$$

hence, $x \in K$ and together with (3.3), we obtain $x \in S_1$. Now it may happen that $\nabla\tilde{\Psi}_m(x) = x$. In this case, by (3.3), we know $x = \nabla\tilde{\Psi}_m(x) \in \partial_{nt}\Lambda$, so in particular $x \in \partial\Lambda \cap \partial V_m$. Moreover, by (3.2), we also have $x \in \partial(\Omega \cap U_m)$. If $x \in \partial U_m \cap \Omega$, then it follows that $x \in \partial V_m \cap \Omega$, a contradiction to the fact that the free boundary does not enter the common region (see Remark 2.16). Therefore, we must have $x \in \partial V_m \cap \partial\Omega$; hence, Lemma 3.5 implies $x \in S_2$ and so $\overline{S_1} \subset S_1 \cup S_2$. \square

Corollary 3.8. *Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then for all $z \in \partial\Lambda$ and $R > 0$ with $\overline{B_R(z)} \cap \partial\Omega = \emptyset$, we have that $\nabla\tilde{\Psi}_m(S_1) \cap \overline{B_R(z)}$ is compact. In particular, $\nabla\tilde{\Psi}_m(S_1) \setminus \partial\Omega$ is relatively closed in $\partial\Lambda \setminus \partial\Omega$.*

Proof. It suffices to prove that $\nabla\tilde{\Psi}_m(S_1) \cap \overline{B_R(z)}$ is closed. Let $y_n \in \nabla\tilde{\Psi}_m(S_1) \cap \overline{B_R(z)}$ and suppose $y_n \rightarrow y \in \partial_{nt}\Lambda \cap \overline{B_R(z)}$. Set $x_n := \nabla\tilde{\Psi}_m^*(y_n)$ and $x := \nabla\tilde{\Psi}_m^*(y)$. Then by repeating the proof of Lemma 3.7, it follows that $x \in \partial(\Omega \cap U_m)$, $L(x) \cap \overline{\Omega \cap U_m} \subset \partial(\Omega \cap U_m)$, and $x \in \nabla\tilde{\Psi}_m^{-1}(\partial_{nt}\Lambda)$. Since $y_n \in \nabla\tilde{\Psi}_m(S_1)$, it also follows from Remark 3.4 that $x_n \in \partial\Omega$; hence, $x \in \partial\Omega$. Now if $y = \nabla\tilde{\Psi}_m(x) = x$, then $y \in \partial\Omega$. However, $y \in \overline{B_R(z)}$, and by assumption, $\overline{B_R(z)} \cap \partial\Omega = \emptyset$. Thus, $\nabla\tilde{\Psi}_m(x) \neq x$, and $y \in \nabla\tilde{\Psi}_m(S_1) \cap \overline{B_R(z)}$. Since $\nabla\tilde{\Psi}_m(S_1) \cap \overline{B_R(z)}$ is compact, $\nabla\tilde{\Psi}_m(S_1) \setminus \partial\Omega$ is relatively closed in $\partial\Lambda \setminus \partial\Omega$. \square

Remark 3.9. *By arguing as in the proof of Corollary 3.8, one may similarly deduce that the set $\nabla \tilde{\Psi}_m(A_1) \setminus \partial\Omega$ is relatively closed in $\partial\Lambda \setminus \partial\Omega$. Moreover, it is not hard to see that S_1 and A_1 are relatively closed in $\partial\Omega \setminus \partial\Lambda$.*

Next, we generalize the uniform localization lemma of Caffarelli and McCann [7, Lemma 7.11] to the case in which the supports may have a nontrivial intersection. Our proof is by contradiction and follows the line of reasoning for the disjoint case although a new ingredient is required to get around the lack of a separating hyperplane. Our key observation is that one may fully identify the exposed points of the closed convex set Z_{min} that shows up in the work of Caffarelli and McCann. Indeed in that context, thanks to the fact that $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$, the authors only need that all exposed points lie in the set $\bar{\Lambda}$ to obtain the contradiction; however, this is not enough in the general case. We get around this difficulty by exploiting the fact that $\nabla \tilde{\Psi}_m(x) = x$ for all $x \in \Lambda \setminus V_m$ (see Theorem 2.12). Consequently, a weaker version of the uniform localization lemma for the disjoint case is established; this paves the way for the next section in which we estimate the Hausdorff dimension of the singular set.

Lemma 3.10. *(Uniform localization: general case) Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12 and X_s the compact set in Lemma 3.7. Then for $R > 0$ there exists $\epsilon_0 > 0$ such that for all $z \in \bar{\Omega} \cap \bar{U}_m$ for which $B_R(z) \cap X_s = \emptyset$ and for all $0 \leq \epsilon \leq \epsilon_0$, we have*

$$Z_\epsilon(z) \subset B_R(z).$$

Proof. Suppose not. Then there exists $R > 0$ such that for all $k \in \mathbb{N}$ there exists $0 < \epsilon_k \leq \frac{1}{k}$ and $z_k \in \bar{\Omega} \cap \bar{U}_m$ satisfying $B_R(z_k) \cap X_s = \emptyset$ and $Z_{\epsilon(k)}(z_k) \not\subset B_R(z_k)$. Since $\bar{\Omega} \cap \bar{U}_m$ is compact, along a subsequence we have $z_k \rightarrow z_\infty \in \bar{\Omega} \cap \bar{U}_m$, with

$$B_R(z_\infty) \cap X_s = \emptyset. \tag{3.4}$$

By translating all the data we may assume $\nabla \tilde{\Psi}_m(z_\infty) = 0$. Since $\tilde{\Psi}_m$ is convex, this implies that $\tilde{\Psi}_m$ is minimized at z_∞ . Now by Theorem 2.12, $\nabla \tilde{\Psi}_m(\mathbb{R}^n) = \bar{\Lambda}$ is bounded and each centered affine section is bounded, thus it follows that the slope $\nu_{\epsilon(k)}(z_k)$ of the affine function defining the set $Z_{\epsilon(k)}(z_k)$ is contained in Λ (indeed, a translate of the affine function defining the section serves as a supporting hyperplane for $\tilde{\Psi}_m$). Therefore, along another subsequence $\nu_{\epsilon(k)}(z_k) \rightarrow \nu_\infty \in \bar{\Lambda}$ and we can apply Theorem 2.22 (Blaschke selection theorem) to conclude that the sets $Z_{\epsilon(k)}(z_k)$ converge locally in Hausdorff distance to a closed convex set Z_∞ . Let $Z_{min} := \{x \in \mathbb{R}^n : \tilde{\Psi}_m(x) = \tilde{\Psi}_m(z_\infty)\}$. By the same exact argument as in [7, Lemma 7.11 (Claim #1)], one derives $Z_\infty \subset Z_{min}$ and that Z_∞ contains a line segment L centered at z_∞ of length $\frac{2R}{\alpha}$, where $\alpha := n^{\frac{3}{2}}$ is the constant from John's Lemma (the idea is that if strict convexity fails at a point, then there must be a segment on which the function is affine). Now by Theorem 2.12, we know $\nabla \tilde{\Psi}_m : \bar{U}_m \cap \bar{\Omega} \rightarrow \bar{V}_m \cap \bar{\Lambda}$

is a homeomorphism; hence, Z_{min} cannot intersect $\overline{U_m \cap \Omega}$ except at the single point z_∞ , which necessarily, must lie on the boundary. Therefore, the set $Z_{min} \setminus \{z_\infty\}$ must lie outside of $\overline{U_m \cap \Omega}$. Next, by the same exact argument as in [7, Lemma 7.11 (Claim #2)] we have that the exposed points of Z_{min} (see Definition 2.10) lie in the support of the Monge-Ampère measure of $\tilde{\Psi}_m$. By Lemma 3.1, this implies that the exposed points of Z_{min} lie in $\overline{\Omega \cap U_m}$ or $\Lambda \setminus V_m$; since $\{z_\infty\} = Z_{min} \cap \overline{\Omega \cap U_m}$ and z_∞ is not an exposed point in Z_{min} (due to the existence of L), we have that all exposed points of Z_{min} lie in $\Lambda \setminus V_m$. Since every extreme point (see Definition 2.11) is a limit of exposed points (by Theorem 2.21), we have that the extreme points of Z_{min} also lie in $\Lambda \setminus V_m$. Next, note that if Z_{min} would contain a whole line, then gradient monotonicity would imply $\nabla \tilde{\Psi}_m \cdot e_1 = 0$, where e_1 is the direction of the line. This however, contradicts $\nabla \tilde{\Psi}_m(\mathbb{R}^n) = \overline{\Lambda}$. Since the closed, convex set Z_{min} does not contain a line, by Minkowski's theorem [11, Theorem 18.5] we have $Z_{min} = \text{conv}[\text{ext}[Z_{min}] + \text{rc}[Z_{min}]]$. Also, since $z_\infty \in Z_{min}$, we have Z_{min} is non-empty, so $\text{ext}[Z_{min}]$ is non-empty. Hence, $z_\infty = \sum t_i(x_i + y_i)$, where $\sum t_i = 1$, $x_i \in \Lambda \setminus V_m$, and $y_i \in \text{rc}[Z_{min}]$. Since the recession cone of a convex set is convex, we have that $y := \sum t_i y_i \in \text{rc}[Z_{min}]$. Moreover, by Theorem 2.12, $\nabla \tilde{\Psi}_m(x) = x$ on $\Lambda \setminus V_m$ and by continuity on $\Lambda \setminus V_m$. Combining this fact with the definition of Z_{min} yields $0 = \nabla \tilde{\Psi}_m(x_i) = x_i$. Note that this also shows 0 to be the only extreme point of Z_{min} , which in turn, implies $z_\infty = y \in \text{rc}[Z_{min}]$. Next, we wish to show

$$z_\infty \neq 0. \quad (3.5)$$

Assume by contradiction that $z_\infty = 0$. Recall $\nabla \tilde{\Psi}_m(z_\infty) = 0$, so in particular $\nabla \tilde{\Psi}_m(z_\infty) = z_\infty$. However, $z_\infty \in \partial(\overline{\Omega \cap U_m})$, and $\nabla \tilde{\Psi}_m : \overline{U_m \cap \Omega} \rightarrow \overline{V_m \cap \Lambda}$ is a homeomorphism; therefore, $z_\infty \in \partial(\overline{\Omega \cap U_m}) \cap \partial(\overline{\Lambda \cap V_m})$. Hence,

$$z_\infty \in ((\partial U_m \cap \partial \Lambda \cap \Omega) \cup (\partial \Omega \cap \partial \Lambda) \cup (\partial V_m \cap \partial \Omega \cap \Lambda)) \cap \{\nabla \tilde{\Psi}_m(z) = z\} = S_2,$$

and this contradicts $B_R(z_\infty) \cap X_s = \emptyset$.

Thus, since $z_\infty \neq 0$ is in $\text{rc}[Z_{min}]$, we have that Z_{min} contains a half-line in the direction z_∞ . Consider a basis for \mathbb{R}^n so that z_∞ parallels the negative x_n axis. Let $z \in \mathbb{R}^n$ be arbitrary. Since $\tilde{\Psi}_m$ is convex,

$$\left\langle \nabla \tilde{\Psi}_m(z) - \nabla \tilde{\Psi}_m(ke_n), \frac{z - ke_n}{|z - ke_n|} \right\rangle \geq 0.$$

By taking the limit as $k \rightarrow \infty$, we obtain $\partial_n \tilde{\Psi}_m(z) \geq 0$. Therefore, it follows that $\nabla \tilde{\Psi}_m(\mathbb{R}^n) = \overline{\Lambda} \subset \{x : x_n \geq 0\}$. This implies

$$0 \in \partial \Lambda, \quad (3.6)$$

and that $\{x_n = 0\}$ is a supporting hyperplane for Λ at 0. In particular, z_∞ is a normal to Λ at 0. Recall that all the extreme points of Z_{min} lie in $\Lambda \setminus V_m$ and since 0 is the only

extreme point of Z_{min} ,

$$0 \in \overline{\Lambda \setminus V_m}. \quad (3.7)$$

However, $z_\infty \in \partial(\overline{\Omega \cap U_m})$ so

$$0 = \nabla \tilde{\Psi}_m(z_\infty) \in \partial(\overline{V_m \cap \Lambda}) = (\partial V_m \cap \Lambda) \cup (\partial V_m \cap \partial \Lambda) \cup \left(\partial \Lambda \setminus \partial(\overline{\Lambda \setminus V_m}) \right).$$

Hence, (3.6) and (3.7) imply $0 \in \partial V_m \cap \partial \Lambda$; in particular, 0 is a free boundary point. Since $\nabla \tilde{\Psi}_m^*(0) = z_\infty$ and z_∞ is a normal to Λ at 0, convexity of Λ implies

$$\langle \nabla \tilde{\Psi}_m^*(0) - 0, y - 0 \rangle \leq 0 \quad \forall y \in \Lambda;$$

therefore,

$$z_\infty \in \nabla \tilde{\Psi}_m^{-1}(\partial_{nt} \Lambda). \quad (3.8)$$

Recall also that $z_\infty \in rc[Z_{min}]$ and $0 \in Z_{min}$ so that in particular, by definition of recession cone, $0 + \frac{z_\infty}{|z_\infty|}t \in Z_{min} \quad \forall t \geq 0$, which is, of course, equivalent to

$$\nabla \tilde{\Psi}_m(z_\infty) + \frac{z_\infty - \nabla \tilde{\Psi}_m(z_\infty)}{|z_\infty - \nabla \tilde{\Psi}_m(z_\infty)|}t \in Z_{min} \quad \forall t \geq 0. \quad (3.9)$$

Since $Z_{min} \cap \overline{\Omega \cap U_m} = \{z_\infty\}$, (3.5), (3.8), and (3.9) imply $z_\infty \in S_1$. This contradicts that $z_\infty \notin X_s$ (i.e. (3.4)). □

Remark 3.11. (*Uniform localization: disjoint case*) If Ω and Λ are separated by a hyperplane and $\tilde{\Psi}_m$ is the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12, then $S_2 = \emptyset$ so that

$$X_s = S_1 \cup S_2 = \nabla \tilde{\Psi}_m^{-1}(\partial_{nt} \Lambda) \cap K \subset X_{nt} := \overline{\Omega \cap U_m} \cap \nabla \tilde{\Psi}_m^{-1}(\partial_{nt} \Lambda).$$

Therefore, we obtain Caffarelli and McCann's uniform localization lemma [7, Lemma 7.11] under a weaker hypothesis: namely, that of replacing X_{nt} by X_s .

Equipped with the general uniform localization lemma and the other tools developed so far, we are now in a position to prove that away from a singular set, $\tilde{\Psi}_m$ will be locally p -uniformly convex (recall Definition 2.8); this in turn will readily yield the Hölder continuity of $\nabla \tilde{\Psi}_m^*$ (see Remark 2.9 and Corollary 3.13). The proof is a direct adaptation of the corresponding proof for the disjoint case (cf. [7, Theorem 7.13]); nevertheless, we have decided to include it in the appendix for the reader's convenience.

Theorem 3.12. *Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Given $x \in \overline{\Omega \cap U_m}$ and $R > 0$ there exists $r = r(R, \epsilon_0) > 0$ (where ϵ_0 is from Lemma 3.10) such that $\tilde{\Psi}_m$ will be p -uniformly convex on $\Omega \cap U_m \cap B_{\frac{r}{2}}(x)$ if $B_{3R}(x)$ is disjoint from the closed set $\overline{\Lambda \setminus V_m} \cup X_s$ and has convex intersection with $\Omega \cap U_m$.*

Corollary 3.13. *Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Consider the closed set*

$$F := \nabla \tilde{\Psi}_m(\partial_{nc} U_m) \cup \nabla \tilde{\Psi}_m(X_s),$$

with X_s as in Lemma 3.7. Then $\tilde{\Psi}_m^ \in C_{loc}^{1,\alpha}(\overline{\Lambda \cap V_m} \setminus F)$, where $\alpha := \frac{1}{p-1}$ and p is as in Theorem 3.12.*

Proof. Let $y \in \overline{\Lambda \cap V_m} \setminus F$ and set $x := \nabla \tilde{\Psi}_m^*(y) \in \overline{\Omega \cap U_m}$. Note that $x \notin \partial_{nc} U_m$, so there exists $\delta_1 = \delta(x) > 0$ such that $B_{\delta_1}(x) \cap (\Omega \cap U_m)$ is convex. Moreover, by Lemma 3.7, $X_s := S_1 \cup S_2$ is compact, and since $x \notin X_s$, there exists $\delta_2 > 0$ such that $B_{\delta_2}(x) \cap X_s = \emptyset$. Let $\delta := \min\{\delta_1, \delta_2\}$. Note that since $B_{\delta_2}(x) \cap X_s = \emptyset$ we have $x \notin \partial(\Omega \cap \Lambda) \cap \partial V_m$; thus, by possibly taking δ smaller we may assume without loss of generality that $B_\delta(x) \cap \overline{\Lambda \setminus V_m} = \emptyset$. Then set $R := \frac{\delta}{3}$ so that by Theorem 3.12, there exists $r = r(R, \epsilon_0)$ (where ϵ_0 is from Lemma 3.10) such that $\tilde{\Psi}_m$ will be p -uniformly convex on $\Omega \cap U_m \cap B_{\frac{r}{2}}(x)$. Since the convexity exponent and constant are universal, it follows that $\tilde{\Psi}_m^* \in C^{1, \frac{1}{p-1}}(\overline{\nabla \tilde{\Psi}_m(\Omega \cap U_m \cap B_{\frac{r}{2}}(x))})$. Now $\nabla \tilde{\Psi}_m(\Omega \cap U_m \cap B_{\frac{r}{2}}(x))$ is relatively open in $\Lambda \cap V_m$ (since $\nabla \tilde{\Psi}_m$ is a homeomorphism), so there exists $s > 0$ such that $\tilde{\Psi}_m^* \in C^{1, \frac{1}{p-1}}(\overline{B_s(y) \cap (\Lambda \cap V_m)})$. \square

Remark 3.14. *If $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\nabla \tilde{\Psi}_m(S_2) = \emptyset$ and so $F \subset \nabla \tilde{\Psi}_m(\partial_{nc} U_m) \cup \partial_{nt} \Lambda$; in particular, we obtain [7, Corollary 7.14].*

For $x \in \partial U_m \cap \Omega$, we know that $\nabla \tilde{\Psi}_m(x) - x$ is parallel to the normal of the free boundary by Theorem 2.13. Combining this fact with Corollary 3.13 enables us to derive $C_{loc}^{1,\alpha}$ regularity of the free boundaries inside the domains.

Corollary 3.15. *(Free boundary regularity inside the domains) The free boundaries $\partial V_m \cap \Lambda$ and $\partial U_m \cap \Omega$ are $C_{loc}^{1,\alpha}$ hypersurfaces away from $\partial(\Omega \cap \Lambda)$ with $\alpha := \frac{1}{p-1}$ and p as in Theorem 3.12.*

Proof. We prove the result only for $\partial V_m \cap \Lambda$ since the argument for $\partial U_m \cap \Omega$ is entirely symmetric. Let $y \in (\partial V_m \cap \Lambda) \setminus \partial(\Omega \cap \Lambda)$; in particular, $y \notin S_2 = \nabla \tilde{\Psi}_m(S_2)$ (see Remark 3.6). Moreover, since $\nabla \tilde{\Psi}_m(S_1) \subset \partial V_m \cap \partial \Lambda$, we also have that $y \notin \nabla \tilde{\Psi}_m(S_1)$. Next, as $y \in \partial V_m \cap \Lambda$, we may apply Proposition 2.15 (free boundary never maps to free boundary) to deduce $x := \nabla \tilde{\Psi}_m^{-1}(y) = \nabla \tilde{\Psi}_m^*(y) \notin \overline{\partial U_m \cap \Omega}$. Therefore, $x \in \partial \Omega \setminus \partial V_m$ and so $y \notin \nabla \tilde{\Psi}_m(\partial_{nc} U_m)$. Hence, $y \notin F$ and Corollary 3.13 implies that $\nabla \tilde{\Psi}_m^*$ is locally $C^{1,\alpha}$ at y . Now thanks to Theorem 2.13, $\nabla \tilde{\Psi}_m^*(y) - y$ is different from 0 and gives the direction of the inward normal to V_m ; hence, this normal is locally Hölder continuous with universal exponent $\alpha > 0$. \square

Corollary 3.15 confirms Figalli's prediction on the regularity of the free boundaries [8, Remark 4.15]. Next, we would like to understand the set F that shows up in Corollary

3.13. Our goal in the next section is to prove that under suitable conditions on the domains Ω and Λ , the free boundaries $\overline{\partial U_m \cap \Omega}$ and $\overline{\partial V_m \cap \Lambda}$ are $C_{loc}^{1,\alpha}$ hypersurfaces away from the common region $\Omega \cap \Lambda$ and up to a “small” singular set contained at the intersection of fixed with free boundary (inside the domains, the result follows from Corollary 3.15).

4 Analysis of the singular set

We initiate our study of the singular set with a density estimate which will be the first step toward proving that the free boundaries are locally $C^{1,\alpha}$ hypersurfaces away from a “small” set.

Proposition 4.1. *Let $\Lambda \subset \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be bounded, strictly convex domains and $\tilde{\Psi}_m$ the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then $\mathcal{H}^{n-1}(\partial_{nt}\Lambda \cap \{y : \nabla \tilde{\Psi}_m^*(y) \neq y\}) = 0$.*

Proof. First, recall that

$$\partial_{nt}\Lambda = \{y \in \partial\Lambda \cap \overline{V_m \cap \Lambda} : \langle \nabla \tilde{\Psi}_m^*(y) - y, y - z \rangle \leq 0, \forall z \in \Lambda\},$$

and let $y \in \partial_{nt}\Lambda \cap \{y : \nabla \tilde{\Psi}_m^*(y) \neq y\}$. Then in particular, $y \in \partial(V_m \cap \Lambda)$ and since $\nabla \tilde{\Psi}_m^*(y) \neq y$, by Lemma 3.5 it follows that $y \in \partial(V_m \cap \Lambda) \setminus \partial(\Omega \cap \Lambda)$. Therefore, there exists $R > 0$ such that $(\overline{\partial V_m \cap \Lambda} \cap B_R(y)) \cap \partial(\Omega \cap \Lambda) = \emptyset$. Let $\nu := \frac{\nabla \tilde{\Psi}_m^*(y) - y}{|\nabla \tilde{\Psi}_m^*(y) - y|}$ and $\pi^\perp := \text{span}\{\nu\}$ and note that by convexity of Λ , ν is a normal to Λ at y . Therefore, $\pi := (\pi^\perp)^\perp$ is a supporting hyperplane to Λ at y and we may assume, without loss of generality, that $y = 0$. By Remark 2.14 and taking $R > 0$ smaller (if necessary), we have that the free boundary $\overline{\partial V_m \cap \Lambda} \cap B_R(0)$ is a C^1 graph over π . Fix $\delta > 0$ and consider the cone $C_\delta := \{z = (\tilde{z}, z_n) : |z_n| \leq \delta|\tilde{z}|\}$. It follows that for $\eta > 0$ small enough,

$$\Lambda \cap V_m \cap B_\eta(0) \subset C_\delta \cap \{x_n < 0\} \cap B_\eta(0).$$

Next, we may take $\delta > 0$ smaller (if necessary) to ensure that for all $\eta > 0$ small enough,

$$|B_\eta(0) \cap C_\delta \cap \{x_n < 0\}| \leq \frac{1}{4}|B_\eta(0)|,$$

see Figure 2. Hence,

$$\limsup_{\eta \rightarrow 0} \frac{|\Lambda \cap V_m \cap B_\eta(0)|}{|B_\eta(0)|} \leq \frac{1}{4}.$$

Now by Figalli [8, Proposition 4.4] (and reverse symmetry), it follows that $\partial(V_m \cap \Lambda)$ is $(n-1)$ -rectifiable. Therefore, by a theorem of Federer [2, Theorem 3.61], \mathcal{H}^{n-1} -almost every point of the free boundary has density $\frac{1}{2}$. Since we showed above that for an arbitrary point in $\partial_{nt}\Lambda \cap \{y : \nabla \tilde{\Psi}_m^*(y) \neq y\} \subset \partial(V_m \cap \Lambda)$, the density is less than $\frac{1}{4}$ (in fact, our proof shows that the density is 0), the result follows. \square

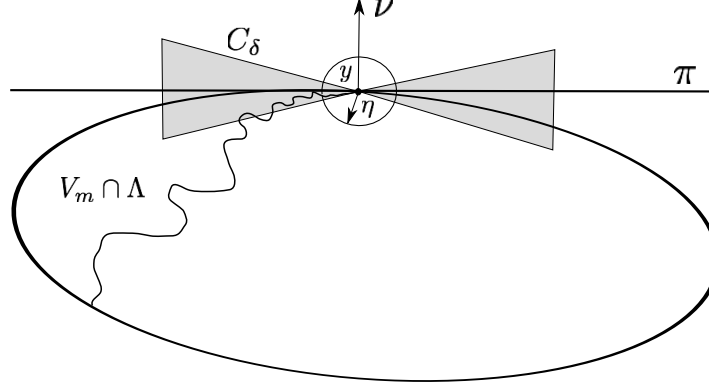


Figure 2: Density estimate.

Next, we apply the above result to the disjoint case from which one can prove that the free boundary $\overline{\partial V_m \cap \Lambda}$ is locally $C^{1,\alpha}$ outside of a set of \mathcal{H}^{n-1} measure zero (an analogous statement holds for $\overline{\partial U_m \cap \Omega}$).

Corollary 4.2. *If $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are strictly convex, bounded domains with $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\mathcal{H}^{n-1}(\partial_{nt}\Lambda) = 0$.*

Proof. This follows directly from the fact that for every $y \in \partial_{nt}\Lambda$, $\nabla \tilde{\Psi}_m^*(y) \in \overline{\Omega}$, so $\nabla \tilde{\Psi}_m^*(y) \neq y$. \square

As seen in Corollary 4.2, Proposition 4.1 immediately implies a result on the Hausdorff dimension of the singular set when the domains are separated by a hyperplane. With some effort, one can obtain this result even in the general case; indeed, this is the content of Corollary 4.12. However, we would like to strengthen this statement by proving that in fact, the free boundaries are locally $C^{1,\alpha}$ outside of an \mathcal{H}^{n-2} σ -finite set. To achieve this task, we need some additional regularity assumptions on the domains and initiate the analysis by developing a method which combines geometric measure theory and convex analysis. The next result is a general statement about projections of convex sets onto other convex sets which will be a crucial tool in our study of the singular set.

Proposition 4.3. *Assume $\Omega \subset \mathbb{R}^n$ is a convex, bounded domain and $\Lambda \subset \mathbb{R}^n$ is a uniformly convex, bounded domain with $C^{1,1}$ boundary. Then*

$$\mathcal{H}^{n-2}(\partial(P_\Lambda(\Omega) \cap \partial\Lambda)) < \infty$$

($\partial P_\Lambda(\Omega)$ is discussed under Definition 2.1).

Proof. If $\partial(P_\Lambda(\Omega) \cap \partial\Lambda) = \emptyset$, then there is nothing to prove (this is the case, for example, if $\Omega \subset \Lambda$). Let $y \in \partial(P_\Lambda(\Omega) \cap \partial\Lambda)$. Since $y \in \partial\Lambda$, convexity of Λ implies the existence of $r_y > 0$ so that $B_{r_y}(y) \cap \partial\Lambda$ may be represented by the graph of a concave function ϕ_y :

$$\Lambda \cap B_{r_y} = \{z \in B_{r_y}(y) : z_n < \phi_y(z_1, \dots, z_{n-1})\}.$$

Without loss of generality, we may assume $\phi_y : B_{r_y}^{n-1}(\tilde{y}) \rightarrow \mathbb{R}$ with $y = (\tilde{y}, \phi(\tilde{y}))$, and $N_\Lambda(y) = (0, 1)$ so that $B_{r_y}^{n-1}(\tilde{y}) \subset \mathbb{T}_y\Lambda - (\tilde{y}, \phi_y(\tilde{y})) \subset \mathbb{R}^{n-1}$ (recall Definition 2.2). Now pick $\delta_y > 0$ with $\delta_y \leq \frac{r_y}{2}$. Let $s_y := \frac{\delta_y}{4}$ so that for all $\tilde{z} \in B_{\frac{\delta_y}{2}}^{n-1}(\tilde{y})$ we have

$$B_{s_y}^{n-1}(\tilde{z}) \subset B_{\frac{3\delta_y}{4}}^{n-1}(\tilde{y}). \quad (4.1)$$

Fix $\tilde{z} \in B_{\frac{\delta_y}{2}}^{n-1}(\tilde{y})$ and set $z := (\tilde{z}, \phi_y(\tilde{z})) \in \partial\Lambda$; there exists $r_z > 0$ such that $\phi_z : B_{r_z}^{n-1}(\tilde{z}) \rightarrow \mathbb{R}$ is a local parameterization of $\partial\Lambda$ at z where $B_{r_z}^{n-1}(\tilde{z}) \subset \mathbb{T}_z\Lambda - (\tilde{z}, \phi_z(\tilde{z}))$ (in this parametrization, $z = (\tilde{z}, \phi_z(\tilde{z}))$). Let $\Phi_y : B_{r_y}^{n-1}(\tilde{y}) \rightarrow \partial\Lambda$ be the map $\Phi_y(\tilde{z}) = (\tilde{z}, \phi(\tilde{z}))$ (Φ_z is similarly defined). Since $(\Phi_y^{-1} \circ \Phi_z)(\tilde{z}) = \tilde{z}$, by continuity of $\Phi_y^{-1} \circ \Phi_z$, we may first pick $\eta = \eta(s_y) > 0$ small enough so that

$$\Phi_y^{-1}(\Phi_z(B_\eta^{n-1}(\tilde{z}))) \subset B_{s_y}^{n-1}(\tilde{z});$$

then by continuity of $\Phi_z^{-1} \circ \Phi_y$, there exists $\mu = \mu(\eta) > 0$ so that

$$B_\mu^{n-1}(\tilde{z}) \subset \Phi_y^{-1}(\Phi_z(B_\eta^{n-1}(\tilde{z}))).$$

Thus, by (4.1) we obtain

$$B_\mu^{n-1}(\tilde{z}) \subset \Phi_y^{-1}(\Phi_z(B_\eta^{n-1}(\tilde{z}))) \subset B_{\frac{3\delta_y}{4}}^{n-1}(\tilde{y}). \quad (4.2)$$

Claim: Let $y \in \partial P_\Lambda(\Omega)$ and $\phi : B_s^{n-1}(\tilde{y}) \rightarrow \mathbb{R}$ be any concave parametrization of $\partial\Lambda$ at $y = (\tilde{y}, \phi(\tilde{y}))$ such that $N_\Lambda(y) = (0, 1)$. Then, there exists an $(n-2)$ -dimensional hyperplane $\pi(\tilde{y})$ and a cone $C_\alpha(\pi(\tilde{y})) \subset \mathbb{R}^{n-1}$ (see Definition 2.3) with $\alpha = \alpha(\Lambda)$ such that

$$\Phi^{-1}(\overline{P_\Lambda(\Omega) \cap \partial\Lambda}) \cap (\tilde{y} + C_\alpha(\pi(\tilde{y}))) = \emptyset,$$

where $\Phi : B_s^{n-1}(\tilde{y}) \rightarrow \partial\Lambda$ is the map $\Phi(\tilde{z}) := (\tilde{z}, \phi(\tilde{z}))$.

Proof of Claim: First, since $\partial\Lambda$ is uniformly convex, there is a constant $C_1 > 0$, such that for all \tilde{x}, \tilde{y} ,

$$\langle \nabla\phi(\tilde{y}) - \nabla\phi(\tilde{x}), \tilde{x} - \tilde{y} \rangle \geq C_1 |\tilde{x} - \tilde{y}|^2. \quad (4.3)$$

Moreover, let $C_2 > 0$ be the uniform Lipschitz constant of $\partial\Lambda$, and

$$x := y + t^*(y)N_\Lambda(y) \in \partial\Omega,$$

where $t^*(y) := \inf\{t \geq 0 : y + tN_\Lambda(y) \in \bar{\Omega}\}$. Since $y \in \partial P_\Lambda(\Omega)$, the half-line $L_t := y + tN_\Lambda(y)$ touches Ω on the boundary at x ; thus, L_t lies on a tangent space of Ω at x with normal $N_\Omega(x)$. This implies $\langle N_\Omega(x), N_\Lambda(y) \rangle = 0$ and since $N_\Lambda(y) = (0, 1)$ we have that $e_{n-1} := N_\Omega(x) \in \mathbb{R}^{n-1}$ (since its n -th component is 0). Next, let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for \mathbb{R}^{n-1} and fix $\tilde{z} \in B_s^{n-1}(\tilde{y})$; thus, $\tilde{z} = \sum_{i=1}^{n-1} b_i e_i + \tilde{y}$ with $\left| \sum_{i=1}^{n-1} b_i e_i \right| \leq s$.

For $\alpha := \frac{C_2}{C_1} > 0$ and $\pi(\tilde{y}) = e_1^\perp = \mathbb{R}^{n-2}$, define

$$C_\alpha(\pi(\tilde{y})) := \{(b_1, \dots, b_{n-1}) = (b_{n-1}^\perp, b_{n-1}) \in \mathbb{R}^{n-1} : \alpha|b_{n-1}^\perp| < b_{n-1}\}.$$

We will now show that $C_\alpha(\pi(\tilde{y}))$ is the desired cone. It suffices to show that if $\tilde{z} \in \tilde{y} + C_\alpha(\pi(\tilde{y}))$, then $\langle N_\Lambda((\tilde{z}, \phi(\tilde{z})), e_{n-1}) \rangle \geq 0$ since if this is true, then for $t \geq 0$,

$$\begin{aligned} \langle z + tN_\Lambda(y) - x, e_{n-1} \rangle &= \langle z + tN_\Lambda(z) - (y + t^*(y)N_\Lambda(y)), N_\Omega(x) \rangle \\ &= \langle \tilde{z} - \tilde{y}, e_{n-1} \rangle + t \langle N_\Lambda(z), e_{n-1} \rangle - t^*(y) \langle N_\Lambda(y), N_\Omega(x) \rangle \\ &\geq b_{n-1} > \alpha|b_{n-1}^\perp| \geq 0, \end{aligned}$$

so by convexity of Ω , $z + tN_\Lambda(y) \notin \bar{\Omega}$, and this implies $\tilde{z} \notin \Phi^{-1}(\overline{P_\Lambda(\Omega) \cap \partial\Lambda})$. Therefore, we will prove that if $\tilde{z} \in \tilde{y} + C_\alpha(\pi(\tilde{y}))$, then $\langle N_\Lambda((\tilde{z}, \phi(\tilde{z})), e_{n-1}) \rangle \geq 0$: since $N_\Lambda(z) = \frac{(-\nabla\phi(\tilde{z}), 1)}{\sqrt{1+|\nabla\phi(\tilde{z})|^2}}$, it suffices to prove $\langle -\nabla\phi(\tilde{z}), e_{n-1} \rangle \geq 0$. Write $b_{n-1}^\perp := \sum_{i=1}^{n-2} b_i e_i$ where

$$\tilde{z} = b_{n-1}^\perp + b_{n-1} e_{n-1} + \tilde{y}.$$

Since $\nabla\phi(\tilde{y}) = 0$, we may use (4.3) and the fact that the Lipschitz constant of $\nabla\phi$ is C_2 to obtain

$$\begin{aligned} \langle \nabla\phi(\tilde{z}), e_{n-1} \rangle &= \langle \nabla\phi(\tilde{z}) - \nabla\phi(\tilde{z} - b_{n-1}^\perp) + \nabla\phi(\tilde{z} - b_{n-1}^\perp) - \nabla\phi(\tilde{y}), e_{n-1} \rangle \\ &\leq C_2|b_{n-1}^\perp| + \frac{1}{b_{n-1}} \langle \nabla\phi(b_{n-1} e_{n-1} + \tilde{y}) - \nabla\phi(\tilde{y}), b_{n-1} e_{n-1} \rangle \\ &\leq C_2|b_{n-1}^\perp| - \frac{1}{b_{n-1}} C_1 |b_{n-1} e_{n-1}|^2 \\ &\leq C_2 \left(\frac{C_1}{C_2} b_{n-1} \right) - C_1 b_{n-1} = 0 \end{aligned}$$

End of Claim.

Let $z \in B_{\frac{\delta_y}{2}}(y) \cap \partial(P_\Lambda(\Omega) \cap \partial\Lambda) \setminus \{y\}$. By the claim we obtain

$$\Phi_z\left(B_\eta^{n-1}(\bar{z}) \cap (\bar{z} + C_\alpha(\pi(\bar{z})))\right) \subset \partial\Lambda \setminus \overline{P_\Lambda(\Omega)}$$

so that by (4.2),

$$\Phi_y^{-1}\left(\Phi_z\left(B_\eta^{n-1}(\bar{z}) \cap (\bar{z} + C_\alpha(\pi(\bar{z})))\right)\right) \subset B_{\frac{3}{4}\delta_y}(\tilde{y}) \cap \Phi_y^{-1}(\partial\Lambda \setminus \overline{P_\Lambda(\Omega)}). \quad (4.4)$$

Now $\Phi_y^{-1}(\Phi_z(\bar{z})) = \tilde{z}$, and since Λ is uniformly Lipschitz, $\Phi_y^{-1} \circ \Phi_z$ has a uniform Lipschitz constant; hence, there exists a cone $C_{\tilde{\alpha}}(\tilde{\pi}(\tilde{z}))$, where $\tilde{\alpha}$ depends only on the Lipschitz constant of Λ and $\tilde{\pi}(\tilde{z})$ is an $(n-2)$ -dimensional hyperplane, for which

$$(\tilde{z} + C_{\tilde{\alpha}}(\tilde{\pi}(\tilde{z}))) \cap \Phi_y^{-1}(\Phi_z(B_\eta^{n-1}(\bar{z}))) \subset \Phi_y^{-1}\left(\Phi_z\left(B_\eta^{n-1}(\bar{z}) \cap (\bar{z} + C_\alpha(\pi(\tilde{y})))\right)\right). \quad (4.5)$$

By (4.2), we obtain

$$(\tilde{z} + C_{\tilde{\alpha}}(\tilde{\pi}(\tilde{z}))) \cap B_\mu^{n-1}(\tilde{z}) \subset (\tilde{z} + C_{\tilde{\alpha}}(\tilde{\pi}(\tilde{z}))) \cap \Phi_y^{-1}(\Phi_z(B_\eta^{n-1}(\bar{z}))),$$

which combines with (4.4), and (4.5) to yield,

$$(\tilde{z} + C_{\tilde{\alpha}}(\tilde{\pi}(\tilde{z}))) \cap B_\mu^{n-1}(\tilde{z}) \subset B_{\frac{3}{4}\delta_y}(\tilde{y}) \cap \Phi_y^{-1}(\partial\Lambda \setminus \overline{P_\Lambda(\Omega)}) \subset \Phi_y^{-1}\left(\overline{B_{r_y}(y)} \cap (\partial\Lambda \setminus \overline{P_\Lambda(\Omega)})\right);$$

hence,

$$(\tilde{z} + C_{\tilde{\alpha}}(\tilde{\pi}(\tilde{z}))) \cap B_\mu^{n-1}(\tilde{z}) \cap \Phi_y^{-1}\left(\overline{B_{\frac{\delta_y}{2}}(y)} \cap \partial(P_\Lambda(\Omega) \cap \partial\Lambda)\right) = \emptyset.$$

Now applying Corollary 2.18, we obtain that $\phi_y^{-1}(\overline{B_{\frac{\delta_y}{2}}(y)} \cap \partial(P_\Lambda(\Omega) \cap \partial\Lambda))$ is finitely $(n-2)$ -rectifiable. Since ϕ_y is bi-Lipschitz with uniform Lipschitz constant, we have

$$\mathcal{H}^{n-2}(\overline{B_{\frac{\delta_y}{2}}(y)} \cap \partial(P_\Lambda(\Omega) \cap \partial\Lambda)) < \infty. \quad (4.6)$$

By compactness, there exists $T \in \mathbb{N}$ and $y_i \in \partial(P_\Lambda(\Omega) \cap \partial\Lambda)$ so that

$$\partial(P_\Lambda(\Omega) \cap \partial\Lambda) \subset \bigcup_{i=1}^T B_{\frac{\delta_{y_i}}{2}}(y_i) \cap \partial(P_\Lambda(\Omega) \cap \partial\Lambda).$$

Thus, an application of (4.6) yields $\mathcal{H}^{n-2}(\partial(P_\Lambda(\Omega) \cap \partial\Lambda)) < \infty$. \square

Note that Proposition 4.3 is a purely geometric result. We will now connect this geometry with the optimal partial transport problem.

Corollary 4.4. *Assume $\Omega \subset \mathbb{R}^n$ is a strictly convex, bounded domain and $\Lambda \subset \mathbb{R}^n$ is a uniformly convex, bounded domain with $C^{1,1}$ boundary. Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then*

$$\nabla \tilde{\Psi}_m(A_2) \subset \partial(P_\Lambda(\Omega) \cap \partial\Lambda), \quad (4.7)$$

with A_2 as in Definition 3.3; in particular,

$$\mathcal{H}^{n-2}(\nabla \tilde{\Psi}_m(A_2)) < \infty.$$

Proof. Let $y = \nabla \tilde{\Psi}_m(x) \in \nabla \tilde{\Psi}_m(A_2)$ and $L_t := \nabla \tilde{\Psi}_m(x) + \frac{x - \nabla \tilde{\Psi}_m(x)}{|x - \nabla \tilde{\Psi}_m(x)|} t$. Since $A_2 \subset S_1$, the half-line $\{L_t\}_{t \geq 0}$ is tangent to the active region. Now since $x \in \partial\Omega \setminus \partial U_m$, it follows that L_t is tangent to Ω at x ; hence, it is on a tangent space to Ω at x . Next, let $z = P_\Lambda(x) \in \partial\Lambda$ (recall that P_Λ is the orthogonal projection operator). Then by the properties of the projection operator (and the convexity of Λ), we know that $x - z$ is parallel to $N_\Lambda(z)$. Since $x \in S_1$, it follows that $\nabla \tilde{\Psi}_m(x) \in \partial_{nt}\Lambda$; in particular, $x - \nabla \tilde{\Psi}_m(x)$ is parallel to $N_\Lambda(\nabla \tilde{\Psi}_m(x))$. Thus, by uniqueness of the projection, it readily follows that $z = \nabla \tilde{\Psi}_m(x) = y$. Now combining $\{L_t\}_{t \geq 0} \subset \mathbb{T}_x\Omega$ and $y = P_\Lambda(x)$ yields $y \in \partial(P_\Lambda(\Omega) \cap \partial\Omega)$. Therefore,

$$\nabla \tilde{\Psi}_m(A_2) \subset \partial(P_\Lambda(\Omega) \cap \partial\Lambda),$$

and the result follows from Proposition 4.3. \square

We will now turn our attention to the set A_1 . Indeed, recall that $S_1 = A_1 \cup A_2$, and as evidenced in Corollary 4.4, the set A_2 has a rich geometric structure. Analogously, the next proposition provides insight into the geometry of A_1 (via Corollary 4.6).

Proposition 4.5. *(Nontransverse intersection points never map to nontransverse intersection points) Suppose that $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, strictly convex domains, and let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then*

$$\nabla \tilde{\Psi}_m(\partial_{nt}\Omega) \cap \partial_{nt}\Lambda = \emptyset,$$

where $\partial_{nt}\Lambda$ (and by duality $\partial_{nt}\Omega$) is defined in (2.3).

Proof. Let

$$\nabla \tilde{\Psi}_m(x) =: y \in \nabla \tilde{\Psi}_m(\partial_{nt}\Omega) \cap \partial_{nt}\Lambda$$

and suppose $\Omega \cap \Lambda \neq \emptyset$. If $x = y$, then by strict convexity,

$$\langle N_\Lambda(x), z - x \rangle < 0$$

for all $z \in \bar{\Lambda}$. However, we also have $N_\Omega(x) = -N_\Lambda(x)$ (since $x = y \in \nabla \tilde{\Psi}_m(\partial_{nt}\Omega) \cap \partial_{nt}\Lambda$) so that

$$\langle N_\Omega(x), z - x \rangle > 0$$

for all $z \in \overline{\Omega}$. Now, pick $z \in \Omega \cap \Lambda$; then from the convexity of Ω we have

$$\langle N_\Omega(x), z - x \rangle \leq 0,$$

a contradiction. Therefore, we may assume without loss of generality that $x \neq y$. By definition of $\partial_{nt}\Omega$ and $\partial_{nt}\Lambda$, $y - x$ is parallel to a normal of Ω at x and $x - y$ is parallel to a normal to Λ at y . Using the strict convexity of Λ and convexity of Ω , this means that for $z \in \Lambda \cap \Omega$,

$$\langle x - y, z - y \rangle < 0,$$

and

$$\langle y - x, z - x \rangle \leq 0.$$

Thus,

$$0 < |x - y|^2 = \langle x - y, x - y \rangle = \langle x - y, x - z \rangle + \langle x - y, z - y \rangle < 0,$$

a contradiction. Therefore, we have reduced the problem to the case when $\Omega \cap \Lambda = \emptyset$. Suppose $x_0 \in \nabla \tilde{\Psi}_m(\partial_{nt}\Omega) \cap \partial_{nt}\Lambda$ and let $x_1 \in \partial U_m \cap \Omega$. By strict convexity of Λ , note that $d := \text{dist}(\nabla \tilde{\Psi}_m(x_1), \mathbb{T}_{\nabla \tilde{\Psi}_m(x_0)}\Lambda) > 0$ and

$$|\nabla \tilde{\Psi}_m(x_0) - x_0| + d \leq |\nabla \tilde{\Psi}_m(x_1) - x_1|. \quad (4.8)$$

By continuity of $\nabla \tilde{\Psi}_m$, for $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\nabla \tilde{\Psi}_m(B_\delta(x_1) \cap U_m) \subset B_\epsilon(\nabla \tilde{\Psi}_m(x_1)) \cap V_m.$$

Now let $A_\delta := B_\delta(x_1) \cap U_m$, and for $\eta > 0$, set $A_\eta := B_\eta(x_0) \cap U_m^c \cap \Omega$. Pick $\frac{d}{2} > \epsilon > 0$ small enough so that $A_\eta \cap A_\delta = \emptyset$ (see Figure 3). Then, by possibly reducing $\epsilon > 0$ (thereby also reducing δ), we may pick $\eta = \eta(\epsilon) > 0$ small so that

$$\int_{A_\eta} f(x) dx = \int_{A_\delta} f_m(x) dx. \quad (4.9)$$

Next, let $\mu = \mu(\epsilon) > 0$ be small enough so that

$$\int_{A_\eta} f(x) dx = \int_{B_\mu(\nabla \tilde{\Psi}_m(x_0)) \cap V_m^c} g(x) dx,$$

and let

$$T_\epsilon : A_\eta \rightarrow B_\mu(\nabla \tilde{\Psi}_m(x_0)) \cap V_m^c \cap \Lambda$$

be the optimal transport map between $f\chi_{A_\eta}$ and $g\chi_{D_\mu}$, where

$$D_\mu := B_\mu(\nabla \tilde{\Psi}_m(x_0)) \cap V_m^c \cap \Lambda.$$

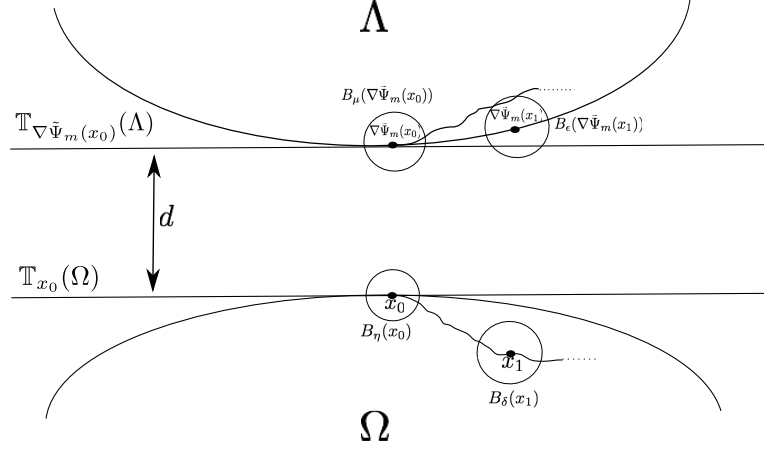


Figure 3: Constructing a cheaper transference plan.

Define

$$\tilde{T}(x) := \begin{cases} T_\epsilon(x), & x \in A_\eta \\ x, & x \in A_\delta \\ \nabla \tilde{\Psi}_m(x), & x \in U_m \setminus B_\delta(x_1), \end{cases}$$

$$\tilde{f}(x) := \begin{cases} f(x), & x \in A_\eta \\ f_m(x), & x \in U_m \setminus B_\delta(x_1) \\ 0, & \text{otherwise.} \end{cases}$$

Set $\tilde{\gamma} := (Id \times \tilde{T})_\# \tilde{f}$; it is easy to check that $\tilde{\gamma}$ is admissible. Now let $z \in A_\eta$ and $w \in A_\delta$ and select ϵ small enough so that

$$\eta(\epsilon) + \mu(\epsilon) + \delta(\epsilon) + \epsilon < \frac{d}{2}.$$

Then, by (4.8) and the triangle inequality we obtain

$$\begin{aligned} |z - T_\epsilon(z)| &\leq |z - x_0| + |x_0 - \nabla \tilde{\Psi}_m(x_0)| + |\nabla \tilde{\Psi}_m(x_0) - T_\epsilon(z)| \\ &\leq \eta(\epsilon) + \mu(\epsilon) + |x_1 - \nabla \tilde{\Psi}_m(x_1)| - d \\ &\leq \eta(\epsilon) + \mu(\epsilon) + |x_1 - w| + |w - \nabla \tilde{\Psi}_m(w)| + |\nabla \tilde{\Psi}_m(w) - \nabla \tilde{\Psi}_m(x_1)| - d \\ &\leq \eta(\epsilon) + \mu(\epsilon) + \delta(\epsilon) + \epsilon - d + |w - \nabla \tilde{\Psi}_m(w)| \\ &\leq |w - \nabla \tilde{\Psi}_m(w)| - \frac{d}{2}. \end{aligned}$$

This shows that the cost of \tilde{T} inside A_η is strictly less than the one of $\nabla\tilde{\Psi}_m$ inside A_δ , and since these maps coincide elsewhere, this contradicts the minimality of $\nabla\tilde{\Psi}_m$. \square

Proposition 4.5 enables us to apply a weak form of the implicit function theorem to prove that A_1 is \mathcal{H}^{n-2} σ -finite; moreover, this information combines with the geometry established in Corollary 4.4 to estimate the size of $\nabla\tilde{\Psi}_m(A_1)$. This is the content of the following two corollaries.

Corollary 4.6. *Assume that $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, strictly convex domains. Then the relatively closed set A_1 (see Remark 3.9) is \mathcal{H}^{n-2} σ -finite. Moreover, if Ω has a C^1 boundary, then there exists $\{x_k\}_{k=1}^\infty \subset A_1$ and $R_k > 0$ such that*

$$A_1 \subset \bigcup_{k=1}^\infty B_{R_k}(x_k), \quad (4.10)$$

with $\mathcal{H}^{n-2}(A_1 \cap B_{R_k}(x_k)) < \infty$. If in addition $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then $\mathcal{H}^{n-2}(A_1) < \infty$.

Proof. Let D_Ω denote the set of differentiability points of $\partial\Omega$ and set

$$A_1^1 := A_1 \cap \partial_{nt}\Omega, \quad A_1^2 := (A_1 \setminus \partial_{nt}\Omega) \cap D_\Omega.$$

If $x \in A_1^1$, then $\nabla\tilde{\Psi}_m(x) \in \partial_{nt}\Lambda$. Therefore, $\nabla\tilde{\Psi}_m(A_1^1) \subset \nabla\tilde{\Psi}_m(\partial_{nt}\Omega) \cap \partial_{nt}\Lambda$, so by Proposition 4.5, $A_1^1 = \emptyset$. Next, since $\Omega \subset \mathbb{R}^n$ is convex, it is well-known that the set of non-differentiability points has co-dimension 2 [1]; thus,

$$\mathcal{H}^{n-2}((A_1 \setminus \partial_{nt}\Omega) \setminus D_\Omega) \leq \mathcal{H}^{n-2}(\partial\Omega \setminus D_\Omega) < \infty.$$

Now let $x \in A_1^2$ so that by Remark 3.4, $x \in (\partial U_m \cap \partial\Omega) \setminus \partial_{nt}\Omega$. Therefore, at x , the free boundary ∂U_m touches the fixed boundary transversally so that $N_{U_m}(x) \neq N_\Omega(x)$ and since x is a differentiability point of Ω , we may apply the weak implicit function theorem (see e.g. [13, Corollary 10.52]) to obtain $R(x) > 0$ such that $\partial U_m \cap \partial\Omega \cap B_{R(x)}(x)$ is contained in an $(n-2)$ -dimensional Lipschitz graph. In particular,

$$\mathcal{H}^{n-2}(\partial U_m \cap \partial\Omega \cap B_{R(x)}(x)) < \infty.$$

Now

$$A_1^2 \subset \bigcup_{x \in A_1^2} B_{R(x)}(x); \quad (4.11)$$

thus there exists $\{x_k\}_{k=1}^\infty \subset A_1^2$ such that

$$A_1^2 = \bigcup_{k=1}^\infty (A_1^2 \cap B_{R_k}(x_k)).$$

Let $E_0 = (A_1 \setminus \partial_{nt}\Omega) \setminus D_\Omega$ and $E_k = A_1^2 \cap B_{R_k}(x_k)$. Then we have

$$A_1 = \bigcup_{k=0}^{\infty} E_k, \quad (4.12)$$

with

$$\mathcal{H}^{n-2}(E_k) < \infty,$$

and this proves the \mathcal{H}^{n-2} σ -finiteness of A_1 . If Ω has a C^1 boundary, then $A_1 = A_1^2$ so that (4.10) follows from (4.11). Furthermore, if $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$, then $S_2 = \emptyset$ and $X_s = S_1$ is compact by Lemma 3.7; this implies that $A_1 = A_1^2$ is compact; thus, using (4.11), we may extract a finite subcover to conclude the proof. \square

Corollary 4.7. *Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, strictly convex domains, and let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then the relatively closed set $\nabla\tilde{\Psi}_m(A_1)$ (see Remark 3.9) is \mathcal{H}^{n-2} σ -finite. Moreover, if Ω has a C^1 boundary, then there exist open sets $F_k \subset \partial(\Lambda \cap V_m)$ (in the subspace topology) such that*

$$\nabla\tilde{\Psi}_m(A_1) \subset \bigcup_{k=1}^{\infty} F_k, \quad (4.13)$$

with $\mathcal{H}^{n-2}(\nabla\tilde{\Psi}_m(A_1) \cap F_k) < \infty$. If in addition $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$, then

$$\mathcal{H}^{n-2}(\nabla\tilde{\Psi}_m(A_1)) < \infty.$$

Proof. If $y \in \nabla\tilde{\Psi}_m(A_1)$, then $y = \nabla\tilde{\Psi}_m(x)$ with $x \in A_1 \subset S_1$; in particular, $y \in \partial_{nt}\Lambda$ so that $x - \nabla\tilde{\Psi}_m(x)$ is parallel to a normal of Λ at $\nabla\tilde{\Psi}_m(x) \in \partial\Lambda$. Hence, $\nabla\tilde{\Psi}_m(x) = P_\Lambda(x)$, so that

$$\nabla\tilde{\Psi}_m(A_1) = P_\Lambda(A_1). \quad (4.14)$$

Now from (4.12) in the proof of Corollary 4.6, $A_1 = \bigcup_{k=0}^{\infty} E_k$, with $\mathcal{H}^{n-2}(E_k) < \infty$, so

$$P_\Lambda(A_1) = \bigcup_{k=0}^{\infty} P_\Lambda(E_k),$$

and since P_Λ is Lipschitz, $\mathcal{H}^{n-2}(P_\Lambda(E_k)) \leq \mathcal{H}^{n-2}(E_k) < \infty$, and this proves that $\nabla\tilde{\Psi}_m(A_1)$ is \mathcal{H}^{n-2} σ -finite. If Ω has a C^1 boundary, then we may use (4.10) to define $F_k := \nabla\tilde{\Psi}_m(B_{R_k}(x_k) \cap \partial(\Omega \cap U_m))$; note that since $\nabla\tilde{\Psi}_m$ is a homeomorphism between the active regions, each F_k is open in $\partial(\Lambda \cap V_m)$. Moreover, thanks to (4.14),

$$\mathcal{H}^{n-2}(\nabla\tilde{\Psi}_m(A_1) \cap F_k) = \mathcal{H}^{n-2}(P_\Lambda(A_1) \cap F_k) \leq \mathcal{H}^{n-2}(A_1 \cap B_{R_k}(x_k)) < \infty,$$

and we obtain (4.13). If in addition $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$, then Corollary 4.6 implies $\mathcal{H}^{n-2}(P_\Lambda(A_1)) \leq \mathcal{H}^{n-2}(A_1) < \infty$. \square

Since $S_1 = A_1 \cup A_2$, we are now in a position to prove that the set $\nabla \tilde{\Psi}_m(S_1)$ is \mathcal{H}^{n-2} σ -finite.

Proposition 4.8. *Assume $\Omega \subset \mathbb{R}^n$ is a strictly convex, bounded domain and $\Lambda \subset \mathbb{R}^n$ is a uniformly convex, bounded domain with $C^{1,1}$ boundary. Let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then the relatively closed set $\nabla \tilde{\Psi}_m(S_1)$ (see Corollary 3.8) is \mathcal{H}^{n-2} σ -finite. If Ω has a C^1 boundary, then there exist open sets $\tilde{F}_k \subset \partial(\Lambda \cap V_m)$ such that*

$$\nabla \tilde{\Psi}_m(S_1) \subset \bigcup_{k=1}^{\infty} \tilde{F}_k, \quad (4.15)$$

with $\mathcal{H}^{n-2}(\nabla \tilde{\Psi}_m(S_1) \cap \tilde{F}_k) < \infty$. If in addition $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$, then

$$\mathcal{H}^{n-2}(\nabla \tilde{\Psi}_m(S_1)) < \infty.$$

Proof. Recall $\nabla \tilde{\Psi}_m(A_2) \subset \partial(P_\Lambda(\Omega) \cap \partial\Lambda)$ (see (4.7)), and let $R > 0$; using compactness, there exists $M = M(R) > 0$ and $\{y_k\}_{k=1}^M \subset \partial(P_\Lambda(\Omega) \cap \partial\Lambda)$ for which

$$\nabla \tilde{\Psi}_m(A_2) \subset \partial(P_\Lambda(\Omega) \cap \partial\Lambda) \subset \bigcup_{k=1}^M B_R(y_k);$$

for $k = 1, \dots, M$, let $\tilde{F}_k := B_R(y_k) \cap \partial(\Lambda \cap V_m)$, and note that since $\nabla \tilde{\Psi}_m(A_2) \subset \partial(\Lambda \cap V_m)$,

$$\nabla \tilde{\Psi}_m(A_2) \subset \bigcup_{k=1}^M \tilde{F}_k.$$

Moreover, $\nabla \tilde{\Psi}_m(A_2) \cap \tilde{F}_k \subset \partial(P_\Lambda(\Omega) \cap \partial\Lambda) \cap \tilde{F}_k$, so that, by Proposition 4.3, we have $\mathcal{H}^{n-2}(\nabla \tilde{\Psi}_m(A_2) \cap \tilde{F}_k) < \infty$. Recalling $S_1 = A_1 \cup A_2$ and setting $\tilde{F}_k := F_{k-M}$ for $k = M+1, M+2, \dots$, the result now follows from Corollary 4.7. \square

Before proving the main result of this section (i.e. Theorem 4.10), we need one more statement about the size of the set consisting of points at the intersection of the target free boundary with fixed boundary that are the image of corresponding points at the intersection of the source free boundary with fixed boundary under the partial transport map.

Proposition 4.9. *Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, strictly convex domains, and let*

$$G := \nabla \tilde{\Psi}_m(\partial U_m \cap \partial\Omega) \cap \partial V_m \cap \partial\Lambda,$$

where $\tilde{\Psi}_m$ is the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then G admits a decomposition $G = G_1 \cup G_2$, where G_1 is relatively closed in $\partial\Lambda \setminus \partial\Omega$ and G_2 is compact with \mathcal{H}^{n-2} finite measure. Moreover,

$$G_1 \subset \bigcup_{k=1}^{\infty} B_{R_k}(y_k), \quad (4.16)$$

with $\mathcal{H}^{n-2}(G_1 \cap B_{R_i}(y_i)) < \infty$. If in addition, $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$, then we also have that G_1 is compact with $\mathcal{H}^{n-2}(G_1) < \infty$.

Proof. Consider the decomposition

$$G = G_1 \cup G_2,$$

with

$$G_1 := (\nabla \tilde{\Psi}_m(\partial U_m \cap \partial\Omega) \setminus \partial_{nt}\Lambda) \cap (\partial V_m \cap \partial\Lambda),$$

and

$$G_2 := \nabla \tilde{\Psi}_m(\partial U_m \cap \partial\Omega) \cap \partial_{nt}\Lambda.$$

Note that G_2 is compact; furthermore, split $G_2 = G_2^1 \cup G_2^2$, with

$$G_2^1 := \nabla \tilde{\Psi}_m(\partial_{nt}\Omega) \cap \partial_{nt}\Lambda,$$

$$G_2^2 := \nabla \tilde{\Psi}_m(\partial U_m \cap \partial\Omega \setminus \partial_{nt}\Omega) \cap \partial_{nt}\Lambda.$$

Using Proposition 4.5, $G_2^1 = \emptyset$. Next, observe that $K := \partial U_m \cap \partial\Omega \cap (\nabla \tilde{\Psi}_m)^{-1}(\partial_{nt}\Lambda)$ is compact and by Proposition 4.5,

$$K = \partial U_m \cap \partial\Omega \cap (\nabla \tilde{\Psi}_m)^{-1}(\partial_{nt}\Lambda) \setminus \partial_{nt}\Omega.$$

Applying the weak implicit function theorem, we have that for all $x \in (\partial U_m \cap \partial\Omega) \setminus \partial_{nt}\Omega$, there exists $R(x) > 0$ such that

$$\mathcal{H}^{n-2}(B_{R(x)}(x) \cap \partial U_m \cap \partial\Omega) < \infty.$$

Therefore, by compactness, there exists $M \in \mathbb{N}$ and $\{x_i\}_{i=1}^M \subset K$ such that

$$K \subset \bigcup_{i=1}^M B_{R(x_i)}(x_i).$$

Furthermore, recall that for $x \in (\nabla \tilde{\Psi}_m)^{-1}(\partial_{nt}\Lambda)$, $\nabla \tilde{\Psi}_m(x) = P_\Lambda(x)$ (see e.g. the proof of Corollary 4.4). Hence,

$$\begin{aligned} \mathcal{H}^{n-2}(G_2) &= \mathcal{H}^{n-2}(\nabla \tilde{\Psi}_m(K)) = \mathcal{H}^{n-2}(P_\Lambda(K)) \\ &\leq \mathcal{H}^{n-2}(K) \leq \sum_{i=1}^M \mathcal{H}^{n-2}(B_{R(x_i)}(x_i) \cap K) < \infty. \end{aligned} \quad (4.17)$$

Now we show that G_1 is \mathcal{H}^{n-1} σ -finite. Indeed, by applying the weak implicit function theorem once more, it follows that for all $y \in G_1 \subset (\partial V_m \cap \partial\Omega) \setminus \partial_{nt}\Lambda$, there exists $R(y) > 0$ such that

$$\mathcal{H}^{n-2}(G_1 \cap B_{R(y)}(y)) < \infty;$$

thus, we can find $\{y_k\}_{k=1}^\infty \subset G_1$ for which

$$G_1 \subset \bigcup_{i=1}^\infty B_{R(y_i)}(y_i),$$

with $\mathcal{H}^{n-2}(G_1 \cap B_{R(y_i)}(y_i)) < \infty$; this combined with (4.17) proves (4.16). Next, assume further that $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$. We claim that G_1 is compact. Indeed, let $y_n \in G_1$ with $y_n \rightarrow y \in \partial V_m \cap \partial\Lambda$. Set $x_n := \nabla \tilde{\Psi}_m^*(y_n)$ and note that by continuity, $x_n \rightarrow x = \nabla \tilde{\Psi}_m^*(y)$. Since $\partial U_m \cap \partial\Omega$ is closed, it follows that $x \in \partial U_m \cap \partial\Omega$, so in particular $x \neq y$ (since $\bar{\Omega} \cap \bar{\Lambda} = \emptyset$). But we already know that $y \in \nabla \tilde{\Psi}_m(\partial U_m \cap \partial\Omega) \cap (\partial V_m \cap \partial\Lambda)$; thus, it remains to show $y \notin \partial_{nt}\Lambda$. If $y \in \partial_{nt}\Lambda$, strict convexity of Λ implies that for all $z \in \bar{\Lambda}$,

$$\langle x - y, z - y \rangle < 0.$$

Since $y_n \notin \partial_{nt}\Lambda$, for each $n \in \mathbb{N}$, there exists $z_n \in \Lambda$ for which

$$\langle x_n - y_n, z_n - y_n \rangle \geq 0.$$

Now since $\bar{\Lambda}$ is compact, up to a subsequence, $z_n \rightarrow z \in \bar{\Lambda}$. Taking limits, it follows that

$$\langle x - y, z - y \rangle \geq 0,$$

a contradiction; hence, $y \notin \partial_{nt}\Lambda$, and so G_1 is compact (a similar argument shows that G_1 is relatively closed in $\partial\Lambda \setminus \partial\Omega$); thus, we may replace the infinite union in (4.16) with a finite one to deduce $\mathcal{H}^{n-2}(G_1) < \infty$. \square

Now we have all the ingredients to prove that the free boundaries are local $C^{1,\alpha}$ hypersurfaces up to an explicit \mathcal{H}^{n-2} σ -finite set, which is relatively closed in $(\partial\Omega \cup \partial\Lambda) \setminus \partial(\Omega \cap \Lambda)$.

Theorem 4.10. *Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, uniformly convex domains with $C^{1,1}$ boundaries, and let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then away from $\partial(\Omega \cap \Lambda)$, the free boundary $\overline{\partial V_m \cap \Lambda}$ is a $C_{loc}^{1,\alpha}$ hypersurface up to the \mathcal{H}^{n-2} σ -finite set:*

$$S := ((\nabla \tilde{\Psi}_m(\partial_{nc} U_m) \cup \nabla \tilde{\Psi}_m(S_1)) \cap \partial V_m \cap \partial\Lambda) \setminus \partial\Omega.$$

Moreover, S is relatively closed in $\partial\Lambda \setminus \partial\Omega$ and there exist open sets $S_i \subset \partial(\Lambda \cap V_m)$ (in the subspace topology) for which

$$S \subset \bigcup_{i=1}^{\infty} S_i, \quad (4.18)$$

with $\mathcal{H}^{n-2}(S \cap S_i) < \infty$. If $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$, then the free boundary $\overline{\partial V_m \cap \Lambda}$ is a $C_{loc}^{1,\alpha}$ hypersurface away from the compact, \mathcal{H}^{n-2} finite set:

$$S_d := (\nabla \tilde{\Psi}_m(\partial_{nc} U_m) \cup \nabla \tilde{\Psi}_m(S_1)) \cap \partial V_m \cap \partial \Lambda.$$

By duality and symmetry, an analogous statement holds for $\overline{\partial U_m \cap \Omega}$.

Proof. Let

$$F := \nabla \tilde{\Psi}_m(\partial_{nc} U_m) \cup \nabla \tilde{\Psi}_m(S_1 \cup S_2)$$

as in Corollary 3.13 so that $\overline{\partial V_m \cap \Lambda}$ is $C_{loc}^{1,\alpha}$ away from F ; now recall that $\nabla \tilde{\Psi}_m(S_2) = S_2 \subset \partial(\Omega \cap \Lambda)$ (Remark 3.6). Hence, the singular set for $\overline{\partial V_m \cap \Lambda}$ away from $\partial(\Omega \cap \Lambda)$ is S . Now let

$$S_{tr} := ((\nabla \tilde{\Psi}_m(\partial_{nc} U_m) \setminus \nabla \tilde{\Psi}_m(S_1)) \cap \partial V_m \cap \partial \Lambda) \setminus \partial(\Omega \cap \Lambda),$$

and note

$$S = S_{tr} \cup (\nabla \tilde{\Psi}_m(S_1) \setminus \partial(\Omega \cap \Lambda)),$$

(indeed, recall that the free boundary never enters the common region: Remark 2.16). For $y \in S_{tr}$, set $x := \nabla \tilde{\Psi}_m^*(y)$; since Ω is convex and $x \in \partial_{nc} U_m$, it follows that $x \notin \partial\Omega \setminus \partial U_m$. Moreover, since free boundary never maps to free boundary (by Proposition 2.15), we also have $x \notin \partial U_m \cap \Omega$, which implies $x \in \partial U_m \cap \partial\Omega$. In particular,

$$S_{tr} \subset G,$$

where G is the set from Proposition 4.9. Therefore,

$$S \subset (G \cup \nabla \tilde{\Psi}_m(S_1)) \setminus \partial(\Omega \cap \Lambda), \quad (4.19)$$

and so combining Proposition 4.8 with Proposition 4.9 yields (4.18). Next, since $\partial_{nc} U_m$ is a closed set, Corollary 3.8 implies that S is relatively closed in $\partial\Lambda \setminus \partial\Omega$. To prove the last part of the theorem, assume $\overline{\Omega} \cap \overline{\Lambda} = \emptyset$. Then, $S = S_d$ is closed, hence, compact; thus, extracting a finite subcover from (4.18) concludes the proof of the theorem. \square

Remark 4.11. In the non-disjoint case, the \mathcal{H}^{n-2} σ -finite singular set S from Theorem 4.10 is not established to be compact. However, note that since it is relatively closed in $\partial\Lambda \setminus \partial\Omega$, it follows that it is not dense in $\partial\Lambda \setminus \partial\Omega$, and this excludes a potential pathological scenario. Indeed, for $z \in \partial\Lambda$ and $R > 0$ such that $\overline{B_R(z)} \cap \partial\Omega = \emptyset$, it follows that $S \cap \overline{B_R(z)}$ is compact; thus, we may replace the countable open cover in (4.18) with a finite subcover to conclude

$$\mathcal{H}^{n-2}(S \cap \overline{B_R(z)}) < \infty.$$

Note that to prove Theorem 4.10, we needed a $C^{1,1}$ regularity assumption on the domains Ω and Λ . This regularity was used to prove Proposition 4.3. However, thanks to the density estimate in Proposition 4.1, we can prove that the singular set is \mathcal{H}^{n-1} negligible with only a convexity assumption on the domains.

Corollary 4.12. *Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, strictly convex domains. Then, the free boundary $\overline{\partial V_m} \cap \Lambda$ is a $C_{loc}^{1,\alpha}$ hypersurface away from the relatively closed set S defined in Theorem 4.10. Moreover, $\mathcal{H}^{n-1}(S) = 0$.*

Proof. Let

$$\tilde{G} := (G \cup \nabla \tilde{\Psi}_m(S_1)) \setminus \partial(\Omega \cap \Lambda),$$

where G is the set in Proposition 4.9. By repeating the proof leading to (4.19) in Theorem 4.10, we obtain $S \subset \tilde{G}$ (note that no regularity assumption other than strict convexity was used to get to (4.19)). Now if $x \in \nabla \tilde{\Psi}_m(S_1) \setminus \partial(\Omega \cap \Lambda)$, then $x \in \partial_{nt}\Lambda \cap \{y : \nabla \tilde{\Psi}_m^*(y) \neq y\}$. Thus, by applying Proposition 4.1, it follows that

$$\mathcal{H}^{n-1}(\nabla \tilde{\Psi}_m(S_1) \setminus \partial(\Omega \cap \Lambda)) = 0.$$

Next, by Proposition 4.9, we know that G is \mathcal{H}^{n-2} σ -finite; in particular, $\mathcal{H}^{n-1}(G) = 0$, and this concludes the proof. \square

Corollary 4.12 immediately implies the main result established by Caffarelli and McCann on the regularity of the free boundaries in the optimal partial transport problem.

Corollary 4.13. *([7, Corollary 7.14]) Assume $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ are bounded, strictly convex domains separated by a hyperplane, and let $\tilde{\Psi}_m$ be the $C^1(\mathbb{R}^n)$ extension of Ψ_m to \mathbb{R}^n given by Theorem 2.12. Then, the free boundary $\overline{\partial V_m} \cap \Lambda$ is a $C_{loc}^{1,\alpha}$ hypersurface away from the closed set*

$$\nabla \tilde{\Psi}_m(\partial_{nc}U_m) \cup \partial_{nt}\Lambda.$$

Proof. Simply note that by the positive separation, $S = S_d$ (S and S_d are defined in the statement of Theorem 4.10), and

$$S_d \subset \nabla \tilde{\Psi}_m(\partial_{nc}U_m) \cup \partial_{nt}\Lambda,$$

so the result follows from Corollary 4.12. \square

5 Open problems

1. In Theorem 4.10, we proved that the free boundary is locally $C^{1,\alpha}$ away from $\partial(\Omega \cap \Lambda)$ and a singular set at the intersection of the free boundary with the fixed boundary. Hence, if one would be able to show that the free boundary stays away from $\partial(\Omega \cap \Lambda)$

inside the supports, then it would follow that the free boundary is locally $C^{1,\alpha}$ inside the supports; indeed, Figalli has already established that the free boundaries are globally Hölder continuous [9, Theorem 1]; thus, one could improve his result by proving that such intersections do not happen and applying Theorem 4.10. Moreover, if one can show that the free boundary stays away from $\partial(\Omega \cap \Lambda)$ altogether, it would also follow that the singular set S of Theorem 4.10 is compact (see Remark 4.11); hence, \mathcal{H}^{n-2} - finite (instead of σ -finite). A counterexample in which the free boundary touches the common region would also be enlightening, indicating that singularities may very well exist.

2. In Corollary 3.13, we proved that the partial transport is locally $C^{1,\alpha}$ away from some singular set F . By using the fact that free boundary never maps to free boundary (except possibly at the intersection of fixed with free boundary), we were able to estimate the Hausdorff dimension of a portion of F , which showed up in the form of S in Theorem 4.10, and since the normal to the free boundary is in the direction of transport, we were able to deduce some regularity on the free boundary. However, the entire singular set of the partial transport is still not quite understood. Indeed, the set $\nabla \tilde{\Psi}_m(\partial_{nc} U_m) \subset S$ emerged in the course of proving that the Monge-Ampère measure associated to the partial transport is a doubling measure, see Lemma 3.2. Perhaps one can improve this lemma by replacing the set $\nabla \tilde{\Psi}_m(\partial_{nc} U_m)$ with the set of points for which the Monge-Ampère does not double affinely. Since the free boundaries are semiconvex [8, Proposition 4.5], one should be able to exploit the geometry to obtain estimates on the Hausdorff dimension of this set; this gets into the regularity theory for the Monge-Ampère equation (up to the boundary) for semiconvex domains.

3. Is the singular set S in Theorem 4.10 empty? In the course of our study, we proved that certain subsets of the singular set were empty (see e.g. Proposition 4.5). Peradventure one can argue similarly to deduce that all of S is indeed empty.

4. In Theorem 4.10, we assumed that Ω and Λ were $C^{1,1}$ and uniformly convex domains. This was utilized in the proof of Proposition 4.3, which is a purely geometric statement about two convex sets. Therefore, a natural line of research would be to reduce the $C^{1,1}$ regularity assumption in Proposition 4.3 (it seems plausible for the statement to be true under only a strict convexity assumption). As an application one could thereby utilize the method we developed in Section 4 to improve Theorem 4.10.

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6 Appendix

Proof of Theorem 3.12. Assume $x \in \overline{\Omega \cap U_m}$ and $R > 0$ is such that

$$B_{3R}(x) \cap (\overline{\Lambda \setminus U_m} \cup X_s) = \emptyset,$$

with $B_{3R}(x) \cap \overline{\Omega \cap U_m}$ convex. Let $z_0 \in \overline{\Omega \cap U_m} \cap B_R(x)$ so that $B_R(z_0) \subset B_{2R}(x)$. Thus,

$$B_R(z_0) \cap (\overline{\Lambda \setminus U_m} \cup X_s) = \emptyset.$$

Note also that $B_R(z_0) \cap \overline{\Omega \cap U_m}$ is convex. Since $\nabla \tilde{\Psi}_m(x) = x$ on $\Lambda \setminus \overline{V_m}$, we have

$$\overline{\partial V_m} \cap \overline{\Lambda} \cap \partial(\Omega \cap \Lambda) \subset (\partial\Omega \cap \partial\Lambda \cap \{\nabla \tilde{\Psi}_m(z) = z\} \cup (\partial V_m \cap \partial\Omega \cap \Lambda)) \subset X_s,$$

and since $B_R(z_0)$ is disjoint from X_s it follows that $z_0 \in X$ (X is defined in Lemma 3.2). By Lemma 3.2, $B_R(z_0)$ forms a doubling neighborhood around z_0 . Now Lemma 3.10 tells us that $\lim_{\epsilon \rightarrow 0} Z_\epsilon(z_0) = \{z_0\}$ in Hausdorff distance. So for $R > 0$ pick $\epsilon_0 > 0$ as in Lemma 3.10 so that

$$Z_{s\epsilon_0}(z_0) \subset B_R(z_0) \quad \forall s \in [0, 1]. \quad (6.1)$$

Note that this analysis was valid for any $z_0 \in \overline{\Omega \cap U_m} \cap B_R(x)$ and the ϵ_0 only depends on R . We use this in the following claim.

Claim: Let $t \in (0, 1)$ so that $\frac{t}{1-t} = n^{\frac{3}{2}}$ and choose any $t_2 \in (t, 1)$. Let $s_0 = s_0(t_2, 1)$ be the constant from Lemma 2.19. Then for all $\epsilon \in (0, \epsilon_0]$, $z_0 \in \overline{\Omega \cap U_m} \cap B_R(x)$, and $z_1 \in \overline{\Omega \cap U_m} \cap B_R(x) \cap \partial Z_\epsilon(z_0)$ we have

$$\tilde{\Psi}_m(z_1) \geq \tilde{\Psi}_m(z_0) + \langle \nabla \tilde{\Psi}_m(z_0), z_1 - z_0 \rangle + \frac{\epsilon}{t} s_0(t_2, 1). \quad (6.2)$$

Proof of claim: Without loss of generality, assume $\nabla \tilde{\Psi}_m(z_0) = 0$ and let $z_t := (1 - t)z_0 + tz_1$. Since $z_1 \in \overline{\Omega \cap U_m} \cap B_R(x) \cap \partial Z_\epsilon(z_0)$, it follows that $z_t \in t \cdot \overline{Z_\epsilon(z_0)}$. We would like to apply Lemma 2.19, so take $z \in \text{spt} M_{\tilde{\Psi}_m} \cap Z_\epsilon(z_0) = \overline{\Omega \cap U_m} \cap Z_\epsilon(z_0)$. By (6.1), we know $Z_\epsilon(z_0) \subset B_R(z_0)$ (pick $s = \frac{\epsilon}{\epsilon_0} \leq 1$) so we have that $z \in B_R(z_0) \subset B_{2R}(x)$. Therefore, $B_R(z) \subset B_{3R}(x)$ and since $B_{3R}(x) \cap X_s = \emptyset$, it follows that $B_R(z) \cap X_s = \emptyset$. Thus, by Lemma 3.10, we have that $Z_{sc}(z) \subset B_R(z)$ for all $s \in [0, 1]$. Note that

$$B_R(z) \cap \text{spt} M_{\tilde{\Psi}_m} = B_R(z) \cap (B_{3R}(x) \cap \overline{\Omega \cap U_m});$$

hence, $B_R(z) \cap \text{spt} M_{\tilde{\Psi}_m}$ is convex and still disjoint from $\overline{\partial V_m \cap \Lambda} \cap \partial(\Omega \cap \Lambda)$ (since it is disjoint from X_s), therefore, by Lemma 3.2, $B_R(z)$ is a doubling neighborhood around z and since $Z_{s\epsilon}(z) \subset B_R(z)$ is a convex body for all $s \in [0, 1]$, we satisfy the doubling assumption of Lemma 2.19 (note that the doubling constant is universal depending only on the initial data). Therefore, we have that if

$$\tilde{z} \in \text{spt} M_{\tilde{\Psi}_m} \cap t_2 \cdot Z_\epsilon(z_0) = \overline{\Omega \cap U_m} \cap t_2 \cdot Z_\epsilon(z_0),$$

then $Z_{s_0\epsilon}(\tilde{z}) \subset Z_\epsilon(z_0)$. Now $z_0, z_1 \in B_R(x) \cap \overline{\Omega \cap U_m}$, and by assumption, $B_{3R}(x) \cap \overline{\Omega \cap U_m}$ is convex (hence, $B_R(x) \cap \overline{\Omega \cap U_m}$ is convex); thus, it follows that $z_t \in \overline{\Omega \cap U_m}$. Hence, since $z_t \in \overline{\Omega \cap U_m} \cap t \cdot \overline{Z_\epsilon(z_0)} \subset \overline{\Omega \cap U_m} \cap t_2 \cdot \overline{Z_\epsilon(z_0)}$, we obtain

$$Z_{s_0\epsilon}(z_t) \subset Z_\epsilon(z_0).$$

In particular, since $z_1 \in \partial Z_\epsilon(z_0)$, z_1 is not an interior point of $Z_{s_0\epsilon}(z_t)$. We also claim that z_0 is not an interior point. Indeed, suppose that this is not the case and let $x := z_0 - z_t$. Since $z_t + x = z_0 \in Z_{s_0\epsilon}(z_t)$, by John's Lemma [4, Lemma 2] we have $z_t - \frac{x}{\alpha} \in Z_{s_0\epsilon}(z_t)$ with $\alpha := n^{\frac{3}{2}}$. But $x = t(z_0 - z_1)$ and $\alpha = \frac{t}{1-t}$ so that $z_t - \frac{x}{\alpha} = z_t - (1-t)(z_0 - z_1) = z_1$, a contradiction to the fact that z_1 is not an interior point of $Z_{s_0\epsilon}(z_t)$. Thus, neither z_0 nor z_1 are interior points of $Z_{s_0\epsilon}(z_t) = \{x : \tilde{\Psi}_m(x) < L_\epsilon(x) := \tilde{\Psi}_m(z_t) + \langle \nu_\epsilon, x - z_t \rangle + s_0\epsilon\}$, so

$$\tilde{\Psi}_m(z_0) \geq L_\epsilon(z_0), \quad (6.3)$$

$$\tilde{\Psi}_m(z_1) \geq L_\epsilon(z_1). \quad (6.4)$$

Now since $\tilde{\Psi}_m$ is convex,

$$\tilde{\Psi}_m(z_0) + \langle \nabla \tilde{\Psi}_m(z_0), w - z_0 \rangle \leq \tilde{\Psi}_m(w) \quad \forall w \in \mathbb{R}^n.$$

But $\nabla \tilde{\Psi}_m(z_0) = 0$ and letting $w = z_t$ we readily obtain $\tilde{\Psi}_m(z_0) \leq \tilde{\Psi}_m(z_t)$. Therefore, by combining this information with (6.3),

$$L_\epsilon(z_t) = \tilde{\Psi}_m(z_t) + \epsilon s_0 \geq \tilde{\Psi}_m(z_0) + \epsilon s_0 \geq L_\epsilon(z_0) + \epsilon s_0.$$

This implies that on the line segment from z_0 to z_t , the slope of L_ϵ is at least $\frac{\epsilon s_0}{|z_t - z_0|}$. In particular,

$$L_\epsilon(z_1) \geq L_\epsilon(z_t) + \left(\frac{\epsilon s_0}{|z_t - z_0|} \right) |z_1 - z_t|.$$

Now, using (6.4) and that $\frac{|z_1 - z_t|}{|z_t - z_0|} = \frac{1-t}{t}$, we obtain

$$\begin{aligned} \tilde{\Psi}_m(z_1) &\geq L_\epsilon(z_1) \geq L_\epsilon(z_t) + \left(\frac{\epsilon s_0}{|z_t - z_0|} \right) |z_1 - z_t| \\ &\geq \tilde{\Psi}_m(z_0) + \epsilon s_0 + (\epsilon s_0) \left(\frac{1-t}{t} \right) = \tilde{\Psi}_m(z_0) + \frac{\epsilon s_0}{t}, \end{aligned}$$

which proves the claim.

End of claim.

Now we are ready to prove the theorem. Let $r = r(R, \epsilon_0)$ be the constant from [7, Lemma A.5] and let $z_0 \neq z_1 \in \Omega \cap U_m \cap B_{\frac{r}{2}}(x)$; using (6.1), we may apply [7, Lemma A.5] to obtain $z_1 \in B_r(z_0) \subset Z_{\epsilon_0}(z_0)$. By Lemma 3.10 and [7, Lemma A.8], we know that $Z_\xi(z_0)$ is continuous in the variable ξ and converges uniformly to z_0 as $\xi \rightarrow 0$. This implies the existence of $\epsilon \in (0, \epsilon_0)$ so that $z_1 \in \partial Z_\epsilon(z_0)$. Now choose any $\bar{t} \in (0, 1)$ and let $s_0(0, \bar{t}) \in (0, 1)$ be the corresponding constant from Lemma 2.19. Observe that by Corollary 2.20, $s < s_0(0, \bar{t})^k$ implies

$$Z_{s\epsilon_0}(z_0) \subset \bar{t}^k \cdot Z_{\epsilon_0}(z_0)$$

for $k \in \mathbb{N}$. Let $s := \frac{\epsilon}{\epsilon_0}$ and note that by the uniform convergence of the sections, up to possibly replacing r with some $\tilde{r} < r$ depending on t_0, ϵ_0 , and the initial data, we may assume $s < s_0(0, \bar{t})$ so that there exists $k \in \mathbb{N}$ for which $\frac{\log(s)}{\log(s_0(0, \bar{t}))} \in [k, k+1)$; in particular, $s < s_0(0, \bar{t})^k$ and since $z_1 \in \partial Z_\epsilon(z_0) = \partial Z_{s\epsilon_0}(z_0)$, it follows that $z_1 \in \bar{t}^k \cdot \overline{Z_{\epsilon_0}(z_0)}$. Hence, $z_1 = (1 - \bar{t}^k)z_0 + \bar{t}^k w$ for some $w \in \overline{Z_{\epsilon_0}(z_0)}$. Moreover,

$$|z_1 - z_0| = \bar{t}^k |w - z_0| \leq \bar{t}^{(\frac{\log(s)}{\log(s_0(0, \bar{t}))} - 1)} |w - z_0| = s^{\frac{\log(\bar{t})}{\log(s_0(0, \bar{t}))}} \frac{|w - z_0|}{\bar{t}}.$$

But by (6.1), $w \in \overline{B_R(z_0)}$ so that $|w - z_0| \leq R$. Hence, using the definition of s ,

$$|z_1 - z_0| \leq \gamma(\bar{t}, \epsilon_0, R) \epsilon^{\frac{\log(\bar{t})}{\log(s_0(0, \bar{t}))}}, \quad (6.5)$$

for some explicit constant $\gamma(\bar{t}, \epsilon_0, R)$. Now the convexity of $\tilde{\Psi}_m$ yields

$$\tilde{\Psi}_m(z_0) \geq \tilde{\Psi}_m(z_1) + \langle \nabla \tilde{\Psi}_m(z_1), z_0 - z_1 \rangle,$$

so that by combining this inequality with (6.2) and using (6.5), we obtain

$$\langle \nabla \tilde{\Psi}_m(z_1) - \nabla \tilde{\Psi}_m(z_0), z_1 - z_0 \rangle \geq \frac{\epsilon}{\bar{t}} s_0(t_2, 1) \geq C |z_1 - z_0|^{\frac{\log(s_0(0, \bar{t}))}{\log(\bar{t})}},$$

where $C = C(\epsilon_0, R, t, \bar{t}, t_2) > 0$. Note that since $r = \beta \epsilon_0^{\frac{n}{2}} R^{1-n}$, by picking ϵ_0 smaller if necessary, we may assume without loss of generality that $|z_1 - z_0| < 1$. Therefore, we may take $p := \max\{\frac{\log(s_0(0, \bar{t}))}{\log(\bar{t})}, 2\}$ as the convexity exponent. \square

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